Plan:

1. Poisson Distribution
2. Binomial Distribution.
3. Inequalities. (Jensen's ineq.; Hölder ineq; Chebychev ineq).

1. Poisson distribution. (discrete random variable).

· at unit time, we expect certain rare events to happen., for example.
(decay of a particle in a pile of particles). (having a defective products when you do production).

· μ = expectation number for the # rare events.

$$P_k = \mathbb{P}(\text{actual number of occurren} = k) = e^{-\mu} \cdot (e^{\mu}\text{'s Taylor expansion's } k\text{-th term})$$
$$= e^{-\mu} \cdot \frac{\mu^k}{k!}$$

(see Boas for a derivation of this). §15.9.

(#4 in Problems 15.9). Say on average, you have 4 phone calls per day. what is the probability that ~~the~~ you have no calls.; what about 5 calls?

Ans: $\mu = 4$. $P_0 = e^{-4} \cdot \frac{\mu^0}{0!} = e^{-4} \cdot 1 = e^{-4}$

$P_5 = e^{-4} \cdot \frac{4^5}{5!}$

(#8) Say there are 100 typos in 40 pages in a magazine. On
(a). how many pages do you expect to find no misprint?

· on average, the misprint rate per page $\mu = \frac{100}{40} = 2.5$.

· $P_0 = e^{-\mu}$ (probability of a single page have no misprint).

· Turn this into a binomial distribution problem:
  · $N = 40$ trials.
  · trial success rate $= P_0$.
  · X : random varible = # of success. (success = no misprint).

$$\mathbb{E}(X) = N \cdot P_0 = 40 \cdot e^{-2.5}$$

(b) On how many pages do you expect to find 2 misprint?

· the probability of a single page to have 2 misprint:

$$P_2 = e^{-\mu} \cdot \frac{\mu^2}{2!} \quad (\mu = 2.5).$$

· $N = 40$ trials, the events (= having 2 misprints in a trial).

happens with probability P_2.

$$\mathbb{E}(\#\text{ of events happens}) = 40 \cdot P_2 = 40 \cdot \left(e^{-2.5} \cdot \frac{2.5^2}{2!}\right)$$

Binomial distribution (counting problem).

permutation number: $P(n,k)$ = # of ways to pick out k balls out of n labelled balls. (①②, ⋯, ⓝ), and lay them in row.

= pick k ~~out~~ of n objects, and order them.

$$= \underbrace{\overset{(n-0)}{n} \cdot \overset{(n-1)}{(n-1)} \cdot (n-2) \cdots \overset{(n-(k-1))}{(n-k+1)}}_{k \text{ terms.}}$$

Combination number: $C(n,k)$ = # of ways to pick ~~out~~ k balls out of n balls, and leave them in a pile

= don't care about the order.

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}.$$

· Setup for binomial:
- given N trials, independent of each other.
- each trial has success rate p.
- random variable X = # of success.

$$\cdot\ \mathbb{P}(X=k) = C(N,k) \cdot p^k \cdot (1-p)^{N-k}. \qquad 0 \le k \le N.$$

$$1 = \mathbb{P}(X=0) + \mathbb{P}(X=1) + \cdots + \mathbb{P}(X=N).$$

$$[p+(1-p)]^N = (1-p)^N + C(N,1)\cdot p^1 \cdot (1-p)^{N-1} + \cdots + p^N.$$

Inequalities:

(1) Jensen inequality: · let X be a random variable valued in $\mathbb{R}$

· let φ be a convex function on $\mathbb{R}$: then.

$$\varphi(\mathbb{E}(X)) \le \mathbb{E}(\varphi(X)).$$

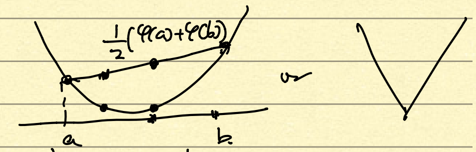
Ex: X satisfies Bernoulli distribution: i.e.

$X = a$ with prob p

$X = b$ with prob $1-p$.

$$\mathbb{E}(X) = a \cdot p + b \cdot (1-p).$$

convex function:



$\forall\ a, b \quad a<b.$

$$\cdot\ \varphi\left(\frac{a+b}{2}\right) \le \frac{1}{2}(\varphi(a)+\varphi(b))$$

$\Leftrightarrow \quad \lambda \in (0,1)$

$$\varphi(\lambda a + (1-\lambda) b) \le \lambda \varphi(a) + (1-\lambda)\varphi(b)$$

$$\varphi(\mathbb{E}(X)) = \varphi(a \cdot p + b \cdot (1-p))$$

$$\mathbb{E}(\varphi(X)) = p \cdot \varphi(a) + (1-p) \cdot \varphi(b)$$

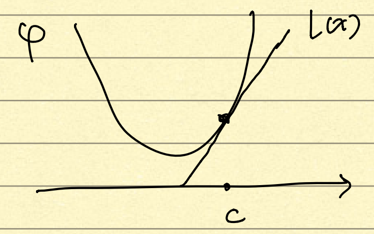
φ Being convex $\Rightarrow$ $\varphi(\mathbb{E}X) \leq \mathbb{E}(\varphi(X))$.

Application: $\varphi(x) = x^2$. $\Rightarrow$ $(\mathbb{E}X)^2 \leq \mathbb{E}(X^2)$.

- define. $c = \mathbb{E}X$. and let $l(x) = ax+b$, such that

$$\begin{cases} l(c) = \varphi(c). \\ l(x) \leq \varphi(x). \end{cases}$$

this is possible, because φ is convex.



- $\mathbb{E}(\varphi(X)) \geq \mathbb{E}(l(X)) = \mathbb{E}(a \cdot X + b) = a \cdot \mathbb{E}(X) + b = a \cdot c + b$

$$= l(\mathbb{E}(X)) = \varphi(\mathbb{E}(X)).$$