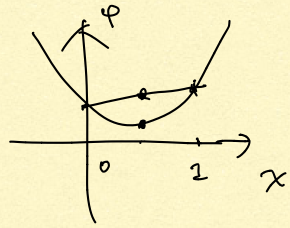
Last time:

(ref: Rick Durrett, Probability Theory & Examples)



- **Jensen's inequality**:

  ① if $\varphi: \mathbb{R} \to \mathbb{R}$ is a convex function.

  X : is a random variable.

  $$\mathbb{E}(\varphi(X)) \geqslant \varphi(\mathbb{E}X).$$

  Hint: try the R.V. X, that values in $\{0, 1\}$. with equal probability $\frac{1}{2}$.

  $\mathbb{E}X = \frac{1}{2}$. $\quad \mathbb{E}(\varphi(X)) = \frac{1}{2} \cdot \varphi(0) + \frac{1}{2}\varphi(1)$

- **Hölder Inequality.** (baby example: Cauchy Inequality).

  - X, Y. random variables:

    $X, Y : \Omega \to \mathbb{R}.$

    (Ω ← sample. space.)

  - let $1 \leqslant p < \infty$. $\quad \|X\|_p = \{\mathbb{E}(|X|^p)\}^{\frac{1}{p}}$

    p-norm

    $p = 2$, $\cdot\ \|(x, y, z)\|_2 = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ length.

    $\cdot$ function f on $[0, 1]$: $\|f\|_p = \left(\int_0^1 |f|^p \, dx\right)^{\frac{1}{p}}$

  - statement:

    $$\mathbb{E}\,|X \cdot Y| \leqslant \|X\|_p \cdot \|Y\|_q.$$

    p, q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

Ex Cauchy ineq: $\quad \mathbb{E}(|X \cdot Y|) \leqslant \|X\|_2 \cdot \|Y\|_2.$

$\Omega = \{1, 2, 3, \cdots, n\}$. $\quad \mathbb{P}$ on Ω is equip-distribution.

$X : \Omega \to \mathbb{R}$. $\quad X(i) = x_i$

$Y(i) = y_i$.

$$\text{LHS} = \mathbb{E}(|X \cdot Y|) = \sum_{i \in \Omega} \mathbb{P}(i) \cdot |X(i)\, Y(i)|.$$

$$= \sum_{i=1}^{n} \cdot \frac{1}{n} |x_i \cdot y_i|.$$

$$\|X\|_2^2 = \mathbb{E}(|X|^2) = \sum_{i=1}^{n} \frac{1}{n} \cdot |x_i|^2,$$

"L^2-norm" of X

$$\|Y\|_2^2 = \sum \frac{1}{n} |y_i|^2.$$

· LHS $\leq$ RHS becomes.

$$\sum_i \frac{1}{n} |x_i y_i| \leq \left(\sum_i \frac{1}{n} |x_i|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_i \frac{1}{n} \cdot |y_i|^2 \right)^{\frac{1}{2}}.$$

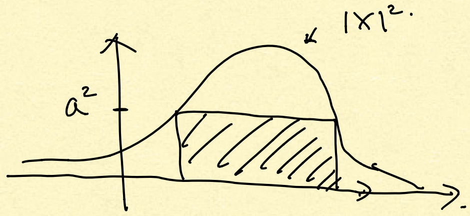
$$\Leftrightarrow \left(\frac{1}{n} \sum_i |x_i y_i| \right)^2 \leq \left(\frac{1}{n} \sum_i |x_i|^2 \right) \cdot \left(\frac{1}{n} \sum_i |y_i|^2 \right).$$

$$\Leftrightarrow \left(\sum_i |x_i y_i| \right)^2 \leq \left(\sum_i |x_i|^2 \right) \left(\sum_i |y_i|^2 \right)$$

· Markov. Inequality: (Chebyshev. inequality).

(1) $X : \Omega \to \mathbb{R}$. , $a > 0$

$$a^2 \, \mathbb{P}(|X| > a) \leq \mathbb{E}(X^2).$$



$$\text{RHS} = \sum_{i \in \Omega} P_i \cdot (X(i))^2 \quad \cdots\cdots ①$$

$$\geq \sum_{i \in A} P_i \, (X(i))^2 \quad \cdots\cdots ②$$

$A \subset \Omega$, $A = \{ i \in \Omega \mid |X(i)| \geq a \}$

$$\geq \sum_{i \in A} P_i \cdot a^2 \quad \cdots\cdots ③$$

$$= a^2 \cdot \mathbb{P}(A)$$

$$= \text{LHS}$$

(2). take any. $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi > 0$. (general statement).

$A \subset \mathbb{R}$ any subset.

· $i_A = \underset{\text{"min"}}{\inf} \{ \varphi(y) : y \in A \}$.

· $\underbrace{i_A \cdot \mathbb{P}(X \in A)}_{③} \leq \underbrace{\mathbb{E}(\varphi(X) : X \in A)}_{②} \leq \underbrace{\mathbb{E}(\varphi(X))}_{①}$