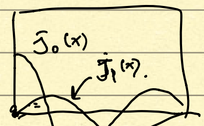
(1) spherical Bessel function.

$$j_n(x) \sim \frac{1}{\sqrt{x}} J_{n+\frac{1}{2}}(x) \sim x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \cdot \left(\frac{\sin x}{x}\right) \quad \left(\frac{\sin x}{x} = j_0(x)\right)$$

$$y_n(x) \sim \frac{1}{\sqrt{x}} Y_{n+\frac{1}{2}}(x). \sim x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \cdot \left(\frac{\cos x}{x}\right) \quad \left(\frac{\cos x}{x} = y_0(x).\right)$$

$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = \left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right) = \frac{\sin x - x\cos x}{x^2} \qquad x \to 0\ ?$$



$$J_p(x) = c\cdot x^p + (\cdot\cdot)x^{p+2} + \cdots$$

$$j_1(x) \sim \frac{1}{\sqrt{x}} J_{1+\frac{1}{2}}(x) \sim \frac{1}{\sqrt{x}}\left(x^{\frac{3}{2}} + \cdots\right)$$

$$\sim \boxed{x^1} + \cdots$$

$$j_n(x) \sim x^n + \cdots$$

$$j_n \sim \frac{1}{\sqrt{x}} J_{n+\frac{1}{2}}(x) \sim x^n + \cdots$$

Hankel functions:

complex valued.

$$\begin{cases} H_n^{(1)}(x) = J_n(x) + i\cdot Y_n(x) \\ H_n^{(2)}(x) = J_n(x) + i\cdot Y_n(x). \end{cases}$$

similar to

$$\begin{cases} e^{ix} = \cos(x) + i\cdot\sin x \\ e^{-ix} = \cdots \end{cases}$$

- $I_n(x)$, $K_n(x)$ Kelvin

  $\sim \underline{J_n(ix)}$

J_n . Y_n

$H_n^{(1)}$, j_n. y_n.

- Airy Function (Ai(x).)

$$y''(x) - x\cdot y(x) = 0$$

"$e^{i\sqrt{-x}x}$" $\quad e^{c\cdot x^{3/2}}$

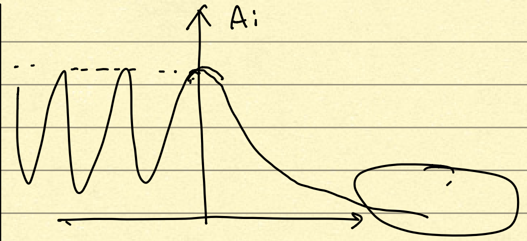
compare with

$$y'' + \omega^2\cdot y(x) = 0 \qquad e^{i\omega x}, e^{-i\omega x}$$

$$\omega^2 \approx -x. \qquad \omega = \sqrt{-x}$$

$x<0$, ω real. we see oscillation

$x>0$, ω imaginary, we see exponential grow / decay.



$$Z = J, Y,$$

$$\underline{A_i(x)} \sim \sqrt{x}\, Z_{1/3}\left(\frac{2}{3}\cdot i\cdot x^{\frac{3}{2}}\right).$$

Exercise:

$$A_i(x) \sim ?\ . \qquad x\to+\infty$$
$$\sim ? \qquad x\to-\infty$$

How to solve Airy Eqn.: $\quad D = \frac{d}{dx}$

$$D^2 y - x\cdot y = 0$$

$$x \leftrightarrow \frac{\partial}{\partial p}.$$

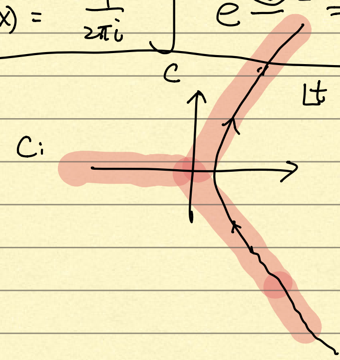
$$\Leftrightarrow \boxed{p^2\cdot\hat{y} - \frac{d}{dp}\cdot\hat{y} = 0} \qquad \hat{y}(p)$$

$$\Rightarrow \hat{y}(p) = e^{p^3}$$

then $\quad y(x) \approx \int \hat{y}(p)\cdot e^{ipx}\, dp$

$$\approx \boxed{\int e^{\frac{1}{3}p^3 + ipx}\, dp} \qquad p = it$$

$$\boxed{y(x) = \frac{1}{2\pi i}\int_C e^{\frac{t^3}{3} \pm t\cdot x}\, dt}$$



Hermite Function (polynomial), Laguerre Function (polynomial)

$$\left(J_n(x) = \sum_{k=0}^{\infty}(\cdots)\left(\frac{x}{2}\right)^{2k+n} \quad \text{not a polynomial} \right)$$

$P_n(x)$ is a degree n polynomial

$$\sim \left(\frac{d}{dx}\right)^n (x^2-1)^n$$

$$y_n'' - x^2 \cdot y_n = -(2n+1)\, y_n.$$

Compare with Harmonic Oscillator.

$$E = p^2 + x^2.$$ more precisely $\left(\frac{p^2}{2m} + x^2.\right)$

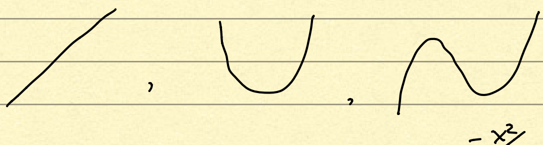
quantize. $p \to \pm i \partial_x$

$$\hat{E} = -\partial_x^2 + x^2.$$

$$\boxed{(-\partial_x^2 + x^2)\, y = \underline{\underline{\lambda}} \cdot y \; .}$$ eigen-value problem.

if $\lambda = 2n+1$, we have a "reasonable" sol'n, i.e. $\int |y|^2\, dx < \infty$

in particular. $|y(x)| \to 0$ as $x \to \infty$



$$y(x) = (\text{polynomial}) \cdot e^{-x^2/2}.$$

$$y_n(x) = (D - x)^n \cdot e^{-x^2/2}.$$ (Hermite functions).

$(D = \frac{d}{dx})$.

$$y_0(x) = e^{-x^2/2}$$
$$y_1(x) = (D - x)\, e^{-x^2/2}$$
$$= (-x - x) \cdot e^{-x^2/2}.$$

$$H_n(x) = (\cdot\cdot)\, e^{\frac{x^2}{2}} \cdot y_n(x)$$
$$= (\pm)\, e^{\frac{x^2}{2}} \cdot (D - x)^n \cdot e^{-\frac{x^2}{2}}.$$
$$= e^{x^2} D^n \cdot e^{-x^2}.$$ (polynomials).

$$\int_{-\infty}^{+\infty} e^{-x^2} \cdot H_n(x)\, H_m(x)\, dx = \begin{cases} 0 & n \neq m \\ C_n & n = m \end{cases}$$

$C_n = \sqrt{\pi} \cdot 2^n \cdot n!$

Aside: orthogonal polynomial.

- input: a "measure" on $\mathbb{R}$

EX "measure" on $\mathbb{R}$

is ① Lebesgue measure dx.

$$(dx)([a,b]) = b - a.$$

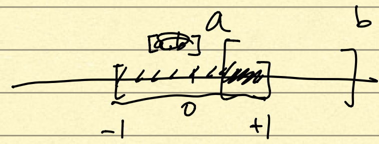
② Gaussian measure: $e^{-x^2} dx$.

- $e^{-x^2} dx\, ([a,b])$

$$= \int_a^b e^{-x^2}\, dx$$

- in general, we donote a measure by $d\mu(x)$.

- another example: (relevat for Legendre function).



$d\mu$ is support on $[-1, 1]$.

$$d\mu([a,b]) = \left| [-1,1] \cap [a,b] \right|$$

Orthogonal polynomial for this measure: $\{F_n(x)\}$

① $F_n(x)$ is a degree n polyn.

② $$\int_{-\infty}^{+\infty} F_n(x)\, F_m(x)\, d\mu(x) = \begin{cases} 0 & n \neq m \\ 1 & n = m. \end{cases}$$

These determines $\{F_n(x)\}$,

$$F_0(x) = C$$

① to solve c,

$$\int c^2\, d\mu = 1.$$

② $F_1(x) = ax + b.$

$$\int F_1(x) \cdot F_0(x)\, d\mu = 0.$$
$$\int (ax + b) \cdot c\, d\mu = 0.$$

$$d\mu(x) = f(x)\, dx$$

$$P_l^m(x) = (1-x^2)^{\frac{m}{2}} \cdot \left(\frac{d}{dx}\right)^{l+m} (x^2-1)^l$$

$P_l^m(\pm 1) = ?$

$$\int_{-1}^{1} P_l^m(x)\, P_n^m(x)\, dx$$

$$= \int_{-1}^{1} \underline{(1-x^2)^m} \cdot \underbrace{\left(\frac{d}{dx}\right)^{l+m} (x^2-1)^l}$$

$$\cdot \left[\left(\frac{d}{dx}\right)^{n+m} (x^2-1)^n \right] dx$$

$x^{\frac{3}{2}} + x^{\cdots}$ $\quad x^{\frac{1}{2}} \quad x^{-\frac{1}{2}}$

13.4

$$\underline{J_{3/2}(x)} = \underline{x^{-1} \cdot (J_{1/2}(x)) - (J_{-1/2}(x))}$$

$$J_{\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1) \cdot \Gamma(n+1+\frac{1}{2})} \cdot \left(\frac{x}{2}\right)^{2n+\frac{1}{2}}$$

$p = \frac{1}{2}$.

$$J_{-\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\, \Gamma(n+1-\frac{1}{2})} \left(\frac{x}{2}\right)^{2n-\frac{1}{2}}$$

$$x^{-1} \cdot J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\, \Gamma(n+1+\frac{1}{2})} \cdot \left(\frac{x}{2}\right)^{2n-\frac{1}{2}} \left(\frac{1}{2}\right)$$

$$\text{RHS} = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^{2n-\frac{1}{2}} \frac{(-1)^n}{\Gamma(n+1)} \left\{ \frac{(1/2)}{\Gamma(n+\frac{1}{2}+1)} - \frac{1}{\Gamma(n+1-\frac{1}{2})} \right\} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (-n)}{n!\ \Gamma(n+\frac{3}{2})}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n-1)!\ \Gamma(n+\frac{3}{2})} \left(\frac{x}{2}\right)^{2n-\frac{1}{2}}$$

for $n=0$. the coeffs should cancel out.

$$\frac{1/2}{\Gamma(\frac{3}{2})} - \frac{1}{\Gamma(\frac{1}{2})} = 0.$$

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2} \cdot \Gamma(\tfrac{1}{2})$$

$$\left(\frac{x}{2}\right)^{2n+\frac{1}{2}} = \left(\frac{x}{2}\right)^{2n-\frac{1}{2}} \cdot \left(\frac{x}{2}\right)$$

$$x^{-1} \cdot \left(\frac{x}{2}\right)^{2n+\frac{1}{2}} = \left(\frac{x}{2}\right)^{2n-\frac{1}{2}} \cdot \left(\frac{1}{2}\right).$$

change $n = m+1$.

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\ \Gamma(m+1+\frac{3}{2})} \left(\frac{x}{2}\right)^{2m+\frac{3}{2}}$$

$$= J_{3/2}(x)$$

$$\text{"} \ \frac{1}{2} - n - 1 + \frac{1}{2}$$

$n > 0$.
$$\frac{(1/2)}{\Gamma(n+1+\frac{1}{2})} - \frac{1}{\Gamma(n+1-\frac{1}{2})} = \frac{(\frac{1}{2}) - (n+1-\frac{1}{2})}{\Gamma(n+1+\frac{1}{2})} = \frac{(-n)}{\Gamma(n+3/2)}$$

$$\Gamma(n+1-\tfrac{1}{2}) = \frac{1}{(n+1-\frac{1}{2})} \Gamma(n+1+\tfrac{1}{2})$$