Poisson Equation:

$$\begin{cases} \Delta u = f \\ \text{B.C.} \quad u(r) \to 0 \text{ as } r \to \infty \end{cases}$$

Laplace Eqn:

$$\Delta u = 0.$$

This Poisson eqn arises in "potential" problems. like gravity / electric field with source term.

Aside

- Schroeding Eqn : not this kind.

$$i \partial_t u(x,t) = -\frac{1}{2} \Delta u + \underbrace{V(x) \cdot u}_{\text{potential term}}$$

How to solve this inhomogeneous equation?

Using Green function to deal with the source term f.

Recall: the Green function $G(\vec{x}, \vec{x}_0)$ solves

$$\begin{cases} \Delta u = \delta(\vec{x} - \vec{x}_0) \\ u(\vec{x}) \to 0 \quad \text{as } |\vec{x}| \to \infty \end{cases}$$

$$\Rightarrow \quad G(\vec{x}, \vec{x}_0) = -\frac{1}{4\pi} \cdot \frac{1}{|\vec{x} - \vec{x}_0|}$$

Then, to solve $\Delta u(\vec{x}) = f(\vec{x})$, we just need to decompose $f(\vec{x})$ as "sum" (integral) of various $\delta(\vec{x} - \vec{x}')$ as $\vec{x}'$ changes. ($\vec{x}'$ as parameters).

$$f(\vec{x}) = \int_{\mathbb{R}^3} \underbrace{f(\vec{x}')}_{\text{as coefficient}} \cdot \delta(\vec{x} - \vec{x}') \, d\vec{x}'$$

Then, the sol'n is the linear combination of $G(\vec{x}, \vec{x}')$ with the same set of coefficients.

$$u(\vec{x}) = \int_{\mathbb{R}^3} f(\vec{x}') \cdot G(\vec{x}, \vec{x}') \cdot d\vec{x}'$$

indeed

(1)
$$\underset{\text{acts on } \vec{x}}{\Delta} u(\vec{x}) = \Delta \int f(x') G(x, x') \cdot dx'$$
$$= \int f(x') \cdot \Delta_x G(x, x') \, dx'$$
$$= \int f(x') \cdot \delta(x - x') \, dx'$$
$$= f(x).$$

u solves the equation

(2) $u(x) \to 0$ as $|x| \to \infty$. because $G(x, x') \to 0$ as $|x| \to \infty$.

and. $f(x)$ has "compact support": vanishes when $|x|$ is large enough.

what if I don't impose B.C. that $u(x) \to 0$ as $|x| \to \infty$.

Do I still have a unique solution to $\Delta u = f$?

- if we replace $u \rightsquigarrow cu = \tilde{u}$. c const. then $\Delta(\tilde{u}) = c \cdot f$ not the same eqn.

- in 1-dim: if $u(x)$ satisfies $\partial_x^2 u(x) = f(x)$.

can you modify $u(x)$, in some way, so it still satisfy the same eqn?

  - adding a constant wouldn't matter
  - adding a linear term would matter.

i.e. adding $(a + bx)$ to u will not break the eqn.

- in 2-dim, suppose $u(x,y)$ satisfy $(\partial_x^2 + \partial_y^2) u(x,y) = f(x,y)$

how many ways can you modify u, without change the eqn?

  - one can add $ax + by + c$ ✓
  - if $\tilde{u} = x^2 + u(x,y)$, then.
$$(\partial_x^2 + \partial_y^2)(\tilde{u}) = \partial_x^2(x^2) + \Delta u$$
$$= 2 + f \quad \text{not the same.}$$
  - if $\tilde{u} = x^2 - y^2 + u(x,y)$.
$$(\partial_x^2 + \partial_y^2)(x^2 - y^2 + u(x,y)) = \Delta(x^2 - y^2) + \Delta u$$
$$= 0 + \Delta u = f. \quad \text{so}$$
$$\Delta \tilde{u} = f. \quad \text{same eqn.}$$
  - if $\tilde{u} = u + v$, and $\Delta \tilde{u} = f$

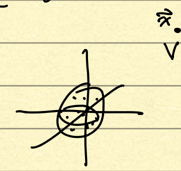
$\Rightarrow \boxed{\Delta v = 0}$ v is a harmonic function.

that is the condition for the modification v to satisfy.

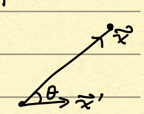
$$\begin{cases} \text{for example: } v = xy. \\ \text{or } v = x^3 - 3xy^2 = \operatorname{Re}(x+iy)^3 \\ \text{or. } v = \operatorname{Re}(\underset{\text{polynomial.}}{P}(x+iy)) \\ \text{they all satisfy } \Delta v = 0. \end{cases}$$

Example : given a charge density $\rho(x,y,z)$ inside the unit ball $|r| < 1$ in $\mathbb{R}^3$

we want to solve for the electric potential at V (



$$V(\vec{x}) = \int \rho(\vec{x}') \cdot G(\vec{x}, \vec{x}') \, d\vec{x}'$$



$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{\sqrt{|\vec{x}|^2 + |\vec{x}'|^2 - 2|\vec{x}| \cdot |\vec{x}'| \cdot \cos\theta}} \qquad \frac{|\vec{x}'|}{|\vec{x}|} \ll 1.$$

$$= \frac{1}{|\vec{x}| \cdot \sqrt{1 + \frac{|\vec{x}'|^2}{|\vec{x}|^2} - 2\frac{|\vec{x}'|}{|\vec{x}|} \cdot \cos\theta}}$$

Recall the generating function of Legendre polynomial

$$\Phi(x,h) = \sum_{n=0}^{\infty} h^n \cdot P_n(x)$$

$$= \frac{1}{\sqrt{1+h^2-2h\cdot x}}$$

$r = |\vec{x}'|$

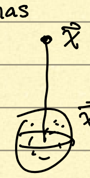
$R = |\vec{x}|$.

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|} \cdot \frac{1}{\sqrt{1+\left(\frac{r}{R}\right)^2 - 2\frac{r}{R}\cdot\cos\theta}}$$

$$= \frac{1}{R} \cdot \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n \cdot P_n(\cos\theta).$$

$$V(\vec{x}) = \int \rho(r,\theta,\phi) \cdot \left(\frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n \cdot P_n(\cos\theta)\right) \cdot \underbrace{r^2 \cdot \sin\theta \cdot dr\, d\theta\, d\phi}_{\text{volume form}}$$

the spherical coordinate is setup.
so that $\vec{x}$ has $\theta = 0$, (ϕ doesn't matter)



Ex 1.3 in Ch 13

- Derive wave equation from the Maxwell equation. (in vacuum)

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = 0 & \text{(1)} \\ \vec{\nabla}\times\vec{E} = -\partial_t \cdot \vec{B} & \text{(2)} \\ \vec{\nabla}\cdot\vec{B} = 0 & \text{(3)} \\ \vec{\nabla}\times\vec{B} = \partial_t \cdot \vec{E} & \text{(4)} \end{cases}$$

← source is zero.

wave equation: $(\partial_t^2 - \vec{\nabla}^2)\, E_i = 0.$ ← scalar

$(\partial_t^2 - \vec{\nabla}^2)\, B_i = 0.$

∂_t (4)

$$\partial_t^2 \vec{E} = \partial_t(\vec{\nabla}\times\vec{B})$$
$$= \vec{\nabla}\times(\partial_t \vec{B})$$
$$= \vec{\nabla}\times(-\vec{\nabla}\times\vec{E}). \quad \text{by (2)}$$
$$= -(\vec{\nabla}\,(\underline{\vec{\nabla}\cdot\vec{E}}) - (\vec{\nabla}\cdot\vec{\nabla})\vec{E})$$
$$= +\Delta\vec{E}.$$

ε_{ijk} symbol. $\varepsilon_{123} = 1$, $\varepsilon_{112} = 0$ ...

$$= \begin{cases} 1 & ijk = \text{cyclic perm of } 1,2,3 \\ -1 & ijk = \text{cyclic -- of } 213. \\ 0 & \text{else repeat indices} \end{cases}$$

$(\vec{A}\times\vec{B})_i = \varepsilon_{ijk} A_j B_k$ sum over j, k.

$$[\vec{A}\times(\vec{B}\times\vec{C})]_i = \varepsilon_{ijk} A_j\, (\vec{B}\times\vec{C})_k$$
$$= \varepsilon_{ijk} A_j\, \varepsilon_{klm} B_l \cdot C_m.$$
$$= \varepsilon_{ijk}\varepsilon_{klm}\, A_j B_l C_m.$$
$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j B_l C_m = \vec{B}\,(\underline{\vec{A}\cdot\vec{C}}) - \vec{C}\cdot(\underline{\vec{A}\cdot\vec{B}})$$

$\varepsilon_{ijk} = \varepsilon_{kij}$, $\sum_k \varepsilon_{kij}\varepsilon_{klm} = \delta_{il}\cdot\delta_{jm} - \delta_{im}\delta_{jl}$