Math 185: Homework 1 Solution

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The following exercises are from Stein's textbook, Chapter 1.

1 Exercise 2

Let $\langle \cdot, \cdot \rangle$ denote the usual product in $\mathbb{R}^2$. In other words, if $Z = (x_1, y_1)$ and $W = (x_2, y_2)$, then

$$\langle Z, W \rangle = x_1 x_2 + y_1 y_2$$

Similarly, we may define a Hermitian inner product $(\cdot, \cdot)$ on $\mathbb{C}$ by

$$(z, w) = z\bar{w}.$$

The term Hermitian is used to describe the fact that $(\cdot, \cdot)$ is not symmetric but rather satisfies the relation

$$(z, w) = \overline{(w, z)}.$$

Show that

$$\langle z, w \rangle = \frac{1}{2}[(z, w) + (w, z)] = \text{Re}(z, w)$$

where we used the usual identification $z = x + iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$.

Solution: Let $z = x + iy$ and $w = u + iv$, then

$$(z, w) = (x + iy)\overline{(u + iv)} = (x + iy)(u - iv) = xu + yv + i(yu - xv)$$

Hence

$$\text{Re}(z, w) = xu + yv = \langle (x, y), (u, v) \rangle = \langle z, w \rangle$$

2 Exercise 7

The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called Blaschke factors, will reappear in various applications in later chapters.

(a) Let z, w be two complex numbers such that $\bar{z}w \neq 1$. Prove that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1$$

and also that

$$\left|\frac{w-z}{1-\bar{w}z}\right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1$$

(b) Prove that for a fixed w in the unit disk $\mathbb{D}$, the mapping

$$F : z \mapsto \frac{w-z}{1-\bar{w}z}$$

satisfies the following conditions:

(i) F maps the unit disk to itself, and is holomorphic.

(ii) F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$.

(iii) $|F(z)| = 1$ if $|z| = 1$.

(iv) $F : \mathbb{D} \to \mathbb{D}$ is bijective.

Solution: (a) Assume $|z| < 1$ and $|w| < 1$. Let $\theta = \arg z$ (or if $z = 0$, let $\theta = 0$), then we define $z' = e^{-i\theta}z$ and $w' = e^{-i\theta}w$. Hence z' is real.

$$z' - w' = e^{-i\theta}(z-w), \quad \bar{w}'z' = \overline{e^{-i\theta}}e^{i\theta}\bar{w}z = \bar{w}z.$$

Thus, we have

$$\frac{w'-z'}{1-\bar{w}'z'} = e^{-i\theta}\frac{w-z}{1-\bar{w}z} \Rightarrow \left|\frac{w'-z'}{1-\bar{w}'z'}\right| = \left|\frac{w-z}{1-\bar{w}z}\right|.$$

Hence, suffice to consider the case where z is replaced by z' and w by w', that is, only consider the case where $z \in \mathbb{R}$ and $z \geq 0$.

Then, it suffices to prove that

$$(r-w)(r-\bar{w}) \leq (1-rw)(1-r\bar{w})$$

for $0 \leq r \leq 1$ and $|w| \leq 1$, with equality achieved if $r = 1$ or $|w| = 1$. Indeed, the above inequality is equivalent to

$$\begin{aligned} &r^2 - r(w+\bar{w}) + |w|^2 \leq 1 - r(w+\bar{w}) + r^2|w|^2 \\ \Leftrightarrow &0 \leq (1-r^2) - (1-r^2)|w|^2 \\ \Leftrightarrow &0 \leq (1-r^2)(1-|w|^2) \end{aligned}$$

Hence we can check that strict inequality is achieve for $r < 1, |w| < 1$, and equality is achieve for $r = 1$ or $|w| = 1$.

(b) (i) F has the desired range, since for $w < 1$ fixed, and $z \in \mathbb{D}$ i.e. $|z| < 1$, we have $|F(z)| < 1$, i.e $F(z) \in \mathbb{D}$ from part (a). Next, we check F is holomorphic. Since $|\bar{w}z| = |w||z| < 1$, we have $1 - \bar{w}z \neq 0$ for $z \in \mathbb{D}$. Thus, using Proposition 2.2 (iii), the fraction $\frac{w-z}{1-\bar{w}z}$ as a function of z is holomorphic for $z \in \mathbb{D}$.

(ii) This is immediate to check.

$$F(0) = \frac{w - 0}{1 - \bar{w}0} = \frac{w}{1} = w,$$

and

$$F(w) = \frac{w - w}{1 - \bar{w}w} = 0.$$

(iii) This follows from part (a).

(iv) We claim that the inverse of F is F, i.e. $F \circ F(z) = z$. Indeed

$$\begin{aligned} F(F(z)) &= \frac{w - \frac{w-z}{1-\bar{w}z}}{1 - \bar{w}\frac{w-z}{1-\bar{w}z}} \\ &= \frac{w(1 - \bar{w}z) - (w - z)}{1 - \bar{w}z - \bar{w}(w - z)} \\ &= \frac{z - |w|^2 z}{1 - |w|^2} \\ &= z. \end{aligned}$$

3 Exercise 16 (a) (c) (e)

Determine the radius of convergence of the series $\sum_{n=1}^{\infty} a_n z^n$ when

(a) $a_n = (\log n)^2$

(c) $a_n = \frac{n^2}{4^n + 3n}$

(e) Find the radius of convergence for the hypergeometric series

$$F(\alpha, \beta, \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n$$

Here $\alpha, \beta \in \mathbb{C}$ and $\gamma \neq 0, -1, -2, \cdots$.

Solution: (a) We have limit

$$(1/n)\log|a_n| = (2/n)\log|\log n| \to 0 \quad \text{as } n \to \infty$$

Hence $1/R = e^0 = 1$, and $R = 1$.

(c) We have limit (better than lim sup)

$$1/R = \lim_{n\to\infty} \left| \frac{n^2}{4^n + 3n} \right|^{1/n} = \frac{\lim_{n\to\infty} |n|^{2/n}}{4 \lim(1 + 3n4^{-n})^{1/n}} = \frac{1}{4}$$

where we used the rules that, if $a = \lim_n a_n, b = \lim_n b_n$, then $\lim_n a_n b_n = ab, \lim_n a_n^{b_n} = a^b$ etc. Hence $R = 4$.

(e) Using ratio test (as justified in exercise 17), we have

$$\frac{a_n}{a_{n-1}} = \frac{(\alpha+n-1)(\beta+n-1)}{n(\gamma+n-1)} = \frac{(1 + \frac{\alpha-1}{n})(1 + \frac{\beta-1}{n})}{(1 + \frac{\gamma-1}{n})} \to 1 \quad \text{as } n \to \infty$$

Hence $R = 1$.

4 Exercise 17

Show that if $\{a_n\}_{n=0}^{\infty}$ is a sequence of non-zero complex numbers such that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$$

then

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

Proof: By assumption, for any $\epsilon > 0$, there exists $N > 0$, such that $\forall n \geq N$,

$$L - \epsilon < \frac{|a_{n+1}|}{|a_n|} < L + \epsilon.$$

Thus, for $n > N$,

$$(L-\epsilon)^{n-N} < \frac{|a_n|}{|a_N|} = \frac{|a_{N+1}|}{|a_N|} \cdots \frac{|a_n|}{|a_{n-1}|} < (L+\epsilon)^{n-N}$$

Multiplying by $|a_N|$ and taking $1/n$-th power, we have

$$(L-\epsilon)^{1-N/n}|a_N|^{1/n} < |a_n|^{1/n} < (L-\epsilon)^{1-N/n}|a_N|^{1/n}.$$

As $n \to \infty$, we have $|a_N|^{1/n} \to 1$ and $(L-\epsilon)^{1-N/n} \to L - \epsilon$, hence we get

$$L - \epsilon < \lim |a_n|^{1/n} < L + \epsilon.$$

Since this is true for any $\epsilon > 0$, we have $\lim |a_n|^{1/n} = L$.

5 Exercise 22

Let $\mathbb{N} = \{1, 2, \cdots, \}$ denote the set of positive integers. A subset $S \subset \mathbb{N}$ is said to be in arithematic progression if

$$S = \{a, a+d, a+2d, \cdots\}$$

for some $a, d \in \mathbb{N}$. Here d is called the step of S. Show that $\mathbb{N}$ cannot be partitioned into finite number of arithematic progressions with distinct step sizes.

Solution: Suppose one can, and let $S_1, \cdots, S_k$ be the collection of arithematic progression, such that $\mathbb{N} = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_k$, with distinct step sizes $d_1, \cdots, d_k$, and offsets a_i. Without loss of generality, assume $d_1 > d_2 > \cdots > d_k > 1$. Then, for $|z| < 1$, we have $\sum_{n\in\mathbb{N}} z^n$ and $\sum_{n\in S_i} z^n$ all absolutely convergent, we thus have

$$\sum_{n\in\mathbb{N}} z^n = \sum_{n\in S_1} z^n + \cdots + \sum_{n\in S_k} z^n$$

Evaluating the sum, we have

$$\frac{z}{1-z} = \frac{z^{a_1}}{1-z^{d_1}} + \cdots + \frac{z^{a_k}}{1-z^{d_k}}, \quad \forall |z| < 1. \tag{*}$$

We claim that this is impossible. Indeed, let $z_0 = e^{2\pi i/d_1}$, then $z_0 \neq 1$, $z_0^{d_1} = 1$, and $z_0^{d_i} \neq 1$ for $i = 2, \cdots, k$. Let $z_n = z_0(1 - 1/n)$ be a sequence of points approaching z_0 within the disk $\mathbb{D}$. Then we see LHS of (*) remains finite, whereas the first term of RHS goes to infinity and other terms of RHS remains finite, which is a contradiction to the equality of (*). Hence it is impossible to have such a partiation of $\mathbb{N}$.