Math 185: Homework 2 Solution

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The following exercises are from Stein's textbook, Chapter 1. 10, 11, 13, 18, 25

Problem (1.10). *Show that*

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \Delta$$

where Δ *is the* **Laplacian**.

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Solution. From page 12 of Stein, we get

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Plug in and you get the desired result.

Remark. Here is a seemingly paradox: we know the intuition of partial derivative, say $\partial/\partial x$ means moving in the direction where only coordinate x is changing. However, what does $\partial/\partial z$ mean, there is no direction where only z change and $\overline{z}$ is fixed. Can you explain?

Problem (1.11). *Use Exercise 10 to show that if f is holomorphic, then the real part and imaginary part of f is harmonic.*

Solution. If f is holomorphic, then $\partial f/\partial \overline{z} = 0$ everywhere, hence further derivative

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} f = 0$$

as well. The same argument can be applied to function $\overline{f}$, which sends z to $\overline{f(z)}$. More precisely, we have

$$\frac{\partial}{\partial z}\overline{f} = \overline{\left(\frac{\partial f}{\partial \overline{z}}\right)} = 0.$$

then apply $\partial/\partial barz$ to it, we get

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}\overline{f} = 0.$$

Thus

$$\Delta \mathrm{Re} f = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}\frac{f+\overline{f}}{2} = 0$$

The case for imaginary part is similar.

Alternatively, you can write $f = u + iv$ for the real and imaginary part, then use Cauchy Riemann condition to get

$$\partial_x^2 u + \partial_y^2 u = \partial_x(\partial_y v) + \partial_y(-\partial_x v) = 0.$$

Remark. Hmm, every holomorphic function has its real part being a harmonic function. Does every harmonic arise in this way? Namely, given a harmonic function u, can we find another harmonic function v, such that $f = u + iv$ is a holomorphic function?

Problem (1.13)**.** *Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:*

1. *Re(f) is constant.*
2. *Im(f) is constant.*
3. *$|f|$ is constant.*

One can conclude f is constant.

Solution. Let $f = u + iv$.

If u is constant, then by Cauchy Riemann condition, we know $\partial_x v = -\partial_y u = 0$ and $\partial_y v = \partial_x u = 0$, hence v is constant. Thus f is constant.

If v is constant, by similar argument, we know u is constant.

If $|f|$ is constant and non-zero, we can say $\mathrm{Re}(\log f)$ is constant, hence $\log f$ is constant. If you complain that we have not learned log, then we can look at problem 9. If you complain that we haven't done problem 9, then we can consider $|f|^2 = u^2 + v^2$ being constant, then

$$0 = \frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}(f\overline{f}) = \frac{\partial f}{\partial z}\frac{\partial \overline{f}}{\partial \overline{z}} = \left|\frac{\partial f}{\partial z}\right|^2 \Rightarrow \frac{\partial f}{\partial z} = 0.$$

This forces f being a constant, thanks to page 23 Corollary 3.4.

Problem (1.18)**.** *Let f be a power series centered at the origin. Prove that f has a power series expansion around any point in its disc of convergence.*

Solution. Say $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is convergent with radius of convergence R. Let $z_0 \in \mathbb{C}$ with $|z_0| < R$. Let u be a complex number, such that $|u| < r = R - |z_0|$, then by triangle inequality

$$|z_0 + u| \leq |z_0| + |u| < R.$$

Hence, for $z = z_0 + u$, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z_0 + u)^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_n \binom{n}{k} z_0^k u^{n-k} \overset{!}{=} \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^k u^{n-k}. \quad (*)$$

The last step is dangerous since it involves a change of order of summations.

Now recall a sufficient condition for switching the order of summation is the following: if $\sum_j \sum_k |a_{jk}| < \infty$, then $\sum_j \sum_k a_{jk} = \sum_k \sum_j a_{jk}$. We are going to use the absolute convergence to show that the double summation is absolutely convergent:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \left| a_n \binom{n}{k} z_0^k u^{n-k} \right| = \sum_{n=0}^{\infty} |a_n| (|z_0| + |u|)^n < \infty$$

where the last step follows from that, the series $\sum_n a_n z^n$ is absolutely convergent for $|z| < R$, and $|z_0| + |u| < R$. Thus we can switch the order of summation in Eq (*).

Thus, for $|u| < r$, we have (introducing $m = n - k$)

$$f(z_0 + u) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_{k+m} \binom{k+m}{k} z_0^k u^m = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} a_{k+m} \binom{k+m}{k} z_0^k u^m$$

where in the above we switched order of summation again, since the double summation is absolute convergent. Introducing $c_m = \sum_{k=0}^{\infty} a_{k+m} \binom{k+m}{k} z_0^k$, we have

$$f(z_0 + u) = \sum_{m=0}^{\infty} c_m u^m \quad \forall |u| < r.$$

Remark. The radius r is not necessarily the radius of convergence for $\sum_m c_m u^m$, because we are estimating c_m by

$$|c_m| \leq \sum_{k=0}^{\infty} |a_{k+m}| \binom{k+m}{k} |z_0|^k$$

ignoring the possible cancellations of the summation k.

Problem (1.25)**.** *The next three calculations provide some insight into Cauchy's theorem, which we treat in the next chapter.*

(a) Evaluate the integrals

$$\int_\gamma z^n dz$$

where $n \in$ and γ is any circle centers at the origin with the positive clockwise orientation

(b) Same question as before, but with γ any circle not containing the origin.

(c) Show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$$

where γ denotes the circle centered at the origin, of radius r, with the positive orientation.

Solution. For (a), we can parameterize $z = re^{i\theta}$ for θ running from 0 to 2π. Then

$$\int_\gamma z^n dz = \int_0^{2\pi} r^n e^{in\theta} re^{i\theta} id\theta = r^{1+n} i \int_0^{2\pi} e^{i(1+n)\theta} d\theta = \begin{cases} 2\pi i & n = -1 \\ 0 & \text{else} \end{cases}$$

Alternatively, for $n \neq -1$, we can find primitive of z^n as $z^{n+1}/(n+1)$ over $\mathbb{C}\backslash\{0\}$, then one can apply Corollary 3.2.

For (b), we parameterize the circle as $z = z_0 + re^{i\theta}$ with $|z_0| > r$. Then again over the cirlce, for $n \neq -1$ we can find primitive of z^n, hence the integral is zero. Suffice to consider the case $n = -1$, thus we have

$$\int_\gamma z^{-1} dz = \int_0^{2\pi} \frac{re^{i\theta}}{z_0 + re^{i\theta}} id\theta$$

Since $|z_0| >$, we can expand the integrand

$$\frac{re^{i\theta}}{z_0 + re^{i\theta}} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{re^{i\theta}}{z_0} \right)^{n+1}$$

Now we are going to switch the order of summation and integration, again, we check the absolute convergence, namely

$$\int_0^{2\pi} \sum_{n=0}^{\infty} \left(\frac{r}{|z_0|} \right)^{n+1} d\theta = 2\pi \frac{r}{|z_0|} \frac{1}{1 - \frac{r}{|z_0|}} < \infty.$$

Hence

$$\int_0^{2\pi} \frac{re^{i\theta}}{z_0 + re^{i\theta}} id\theta = \sum_{n=0} (-1)^n \int_0^{2\pi} \left(\frac{re^{i\theta}}{z_0} \right)^{n+1} d\theta = 0$$

Finally, for (c). We can write

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b} \left(\frac{1}{(z-a)} - \frac{1}{(z-b)} \right),$$

and do integration for both terms. The first term is like (a) where the point a is within the circle $|z| = r$, the second term is like (b) and contribution is zero.

The integral for the first term can be computered using power series again

$$\int_{|z|=r} \frac{1}{z-a} dz = \int_{|z|=r} \frac{1}{z(1-a/z)} dz = \int_{|z|=r} \frac{1}{z(1-a/z)} = \int_{|z|=r} z^{-1}(1+a/z+(a/z)^2+\cdots)dz$$

$$= \sum_{n=0}^{\infty} \int_{|z|=r} z^{-1}(a/z)^{-n} dz = \int_{|z|=r} z^{-1} dz = 2\pi i$$

where when we switch the summation and integral, we again checked that the double sum (more precisely, the integral-sum, is absolutely convergent, meaning if we take the absolute value of the summand-integrand, the integral is still finite).