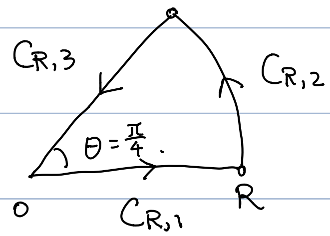
Problems : 1, 2, 4, 5, 6 from Stein. Ch 2.

#1. Prove that $\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$

Proof: Consider the contour integral $\int_{C_R} e^{-z^2}\,dz$, where C_R is as follows



• Since e^{-z^2} is an entire function, by Cauchy theorem, we have $\int_{C_R} e^{-z^2}\, dz = 0.$

- $I_1 = \int_{C_{R,1}} e^{-z^2}\,dz = \int_0^R e^{-x^2}\,dx$

$I_2 = \int_{C_{R,2}} e^{-z^2}\,dz = \int_{\theta=0}^{\pi/4} e^{-(Re^{i})} \cdot d(Re^{i\theta})$

$= \int_{\theta=0}^{\pi/4} e^{-R^2 e^{2i\theta}}\, Re^{i\theta} \cdot i\,d\theta$

$I_3 = \int_{C_{R,3}} e^{-z^2}\,dz = \int_{r=R}^{0} e^{-(re^{i\frac{\pi}{4}})^2}\, d(r\cdot e^{i\frac{\pi}{4}})$

$= -e^{i\frac{\pi}{4}} \int_0^R e^{-ir^2}\,dr = -e^{i\frac{\pi}{4}} \cdot \int_0^R \cos(r^2) - i\sin(r^2)\,dr$

We claim that : $I_2 \to 0$ as $R \to \infty$. Given the claim, we have

$\int_0^\infty \cos(r^2) - i \cdot \sin(r^2)\,dr = \lim_{R\to\infty}\left(\frac{I_3}{-e^{i\pi/4}}\right) = \lim_{R\to\infty} \frac{-I_1 - I_2}{-e^{i\pi/4}}$

$$= \lim_{R\to\infty} \frac{I_1}{e^{i\pi/4}} = e^{-i\frac{\pi}{4}} \cdot \frac{\sqrt{\pi}}{2} = \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) \cdot \frac{\sqrt{\pi}}{2} = \frac{\sqrt{2\pi}}{4} - i\frac{\sqrt{2\pi}}{4}.$$

Since $\int_0^\infty \cos r^2 \, dr$ and $\int_0^\infty \sin(r^2)\, dr$ are both real, we can compare the real and imaginary parts of the above equation and get the desired result.

Now, we turn back to prove the claim.

$$|I_2| = \left| \int_{\theta=0}^{\pi/4} e^{-R^2 e^{2i\theta}} Re^{i\theta} i \cdot d\theta \right|$$

$$\leq \int_0^{\pi/4} e^{-R^2\cos 2\theta} R \cdot d\theta$$

For $\theta \in [0, \pi/4]$, let $\theta = \frac{\pi}{4} - u$, we have

$$\cos(2\theta) = \cos\left(\frac{\pi}{2} - 2u\right) = \sin(2u) \geq \frac{4}{\pi} u \quad \text{for } u \in \left[0, \frac{\pi}{4}\right].$$

$$\int_0^{\frac{\pi}{4}} e^{-R^2\cos 2\theta} d\theta = \int_0^{\frac{\pi}{4}} e^{-R^2 \sin(2u)} du \leq \int_0^{\frac{\pi}{4}} e^{-R^2 \cdot \frac{4}{\pi} u} du$$

$$\leq \int_0^\infty e^{-R^2 \frac{4}{\pi} \cdot u} du = \frac{\pi}{4R^2}.$$

Thus. $|I_2| \leq R \cdot \frac{\pi}{4R^2} = \frac{\pi}{4} \cdot \frac{1}{R} \to 0$ as $R \to \infty$. #.

#2 Show that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$

Proof: $\int_0^\infty \frac{\sin x}{x} dx = \lim_{\substack{\varepsilon\to 0,\\ R\to\infty}} \int_\varepsilon^R \frac{\sin x}{x} dx = \lim_{\cdots} \int_\varepsilon^R \frac{e^{ix} - e^{-ix}}{2ix} dx$

$$= \lim_{\cdots} \left(\int_\varepsilon^R \frac{e^{ix}}{2ix} dx + \int_\varepsilon^R \frac{-e^{-ix}}{2ix} dx \right)$$

let $x = -u$, then $u = -x$ from $-\varepsilon$ to $-R$

$$= \lim \left(\int_{\varepsilon}^{R} \frac{e^{ix}}{2ix}\, dx + \int_{-\varepsilon}^{-R} \frac{-e^{iu}}{-2iu}\, d(-u) \right)$$

switch direction of integration gives a minus sign.

$$= \lim \left(\int_{\varepsilon}^{R} \frac{e^{ix}}{2ix}\, dx + \int_{-R}^{-\varepsilon} \frac{e^{iu}}{2iu}\, du \right)$$

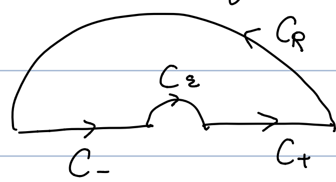
$$= \lim \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) \frac{e^{ix}}{2ix}\, dx \quad - - - - - - \quad (*)$$

Note that, $\int_{\varepsilon}^{R} \frac{1}{x}\, dx = -\int_{-R}^{-\varepsilon} \frac{1}{x}\, dx$, hence the above integral is also

$$\lim_{\cdots} \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) \frac{e^{ix}-1}{2ix}\, dx. \quad \cdots\cdots \quad (**).$$

as suggested by the hint.

In the following, I will give 2 solutions using (*) or (**).

Using (*): Consider the contour 

Then

$$\int_{C_\varepsilon} \frac{e^{iz}}{2iz}\, dz = \int_{C_\varepsilon} \frac{e^{iz}-1}{2iz} + \frac{1}{2iz}\, dz$$

since $(e^{iz}-1)/(2iz)$ is a bounded function near $z=0$,

$\int_{C_\varepsilon} \frac{e^{iz}-1}{2iz}\, dz \to 0$ as $\varepsilon \to 0$.

$$\int_{C_\varepsilon} \frac{1}{2iz}\, dz = \frac{1}{2i} \int_{\theta=\pi}^{0} \frac{1}{\varepsilon e^{i\theta}}\, \varepsilon \cdot e^{i\theta} \cdot i\, d\theta = \frac{1}{2i}(-\pi i) = -\frac{\pi}{2}.$$

$$\int_{C_R} \frac{e^{iz}}{2iz}\, dz \to 0 \quad \text{as} \quad R \to \infty \quad \text{by Jordan Lemma.}$$

Thus

$$\lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \left(\int_{C_-} + \int_{C_+} \right) \frac{e^{iz}}{2iz}\, dz = -\lim_{\cdots} \left(\int_{C_\varepsilon} + \int_{C_R} \right) \frac{e^{iz}}{2iz}\, dz = \frac{\pi}{2}.$$

Using (**): Consider the same contour, now we have.

$$\int_{C_\varepsilon} \frac{e^{iz}-1}{2iz}\,dz \to 0 \quad \text{as} \quad \varepsilon \to 0$$

$$\int_{C_R} \frac{e^{iz}-1}{2iz}\,dz = \int_{C_R} \frac{-1}{2iz}\,dz + \int_{C_R} \frac{e^{iz}}{2iz}\,dz$$

$$\to \int_{C_R} \frac{-1}{2iz}\,dz \qquad \text{as } R \to \infty$$

$$= \frac{-1}{2i} \cdot (\pi i) = -\frac{\pi}{2}.$$

Thus

$$\lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \left(\int_{C_-} + \int_{C_+} \right) \frac{e^{iz}-1}{2iz}\,dz = -\lim_{\cdots} \left(\int_{C_\varepsilon} + \int_{C_R} \right) \frac{e^{iz}-1}{2iz}\,dz = \frac{\pi}{2}.$$

#.

#4 Prove that for all $\xi \in \mathbb{C}$, we have

$$e^{-\pi \cdot \xi^2} = \int_{-\infty}^{+\infty} e^{-\pi x^2} \cdot e^{2\pi i \cdot x \cdot \xi}\,dx.$$

Pf: Consider the RHS of the equation:

$$\int_{-\infty}^{+\infty} e^{-\pi(x^2 - 2ix\xi)}\,dx = \int_{-\infty}^{+\infty} e^{-\pi\left((x-i\xi)^2 - (i\xi)^2\right)}\,dx$$

$$= \int_{-\infty}^{+\infty} e^{-\pi(x-i\xi)^2 - \pi\xi^2}\,dx.$$

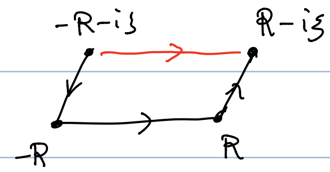
Hence, suffice to prove that

$$\int_{-\infty}^{+\infty} e^{-\pi(x-i\xi)^2}\, dx = 1.$$

Now $\displaystyle\lim_{R\to\infty} \int_{-R}^{R} e^{-\pi(x-i\xi)^2}\, dx = \lim_{R\to\infty} \int_{-R-i\xi}^{R-i\xi} e^{-\pi u^2}\, du.$

where $u = x - i\xi$.

$$= \lim_{R\to\infty} \left(\int_{-R-i\xi}^{-R} + \int_{-R}^{R} + \int_{R}^{R-i\xi} \right) e^{-\pi u^2}\, du$$



Suffice to prove that $\displaystyle\int_{-R-i\xi}^{-R} e^{-\pi u^2}\, du \to 0$ and $\displaystyle\int_{R}^{R-i\xi} e^{-\pi u^2}\, du \to 0.$

$$\left| \int_{R}^{R-i\xi} e^{-\pi u^2}\, du \right| = \left| \int_{t=0}^{1} e^{-\pi(R-i\xi t)^2}\, d(R - i\xi t) \right|$$

$$|e^z| = e^{Re(z)}$$

$$\leq \int_0^1 e^{-\pi \cdot Re(R-i\xi t)^2}\, |\xi| \cdot dt$$

$$\leq \left(\max_{t\in[0,1]} e^{-\pi \cdot Re((R-i\xi t)^2)} \right) \cdot |\xi|$$

$\because$ $Re((R-i\xi t)^2) = Re(R^2 - 2i\xi t \cdot R + (i\xi t)^2)$

$$\geq R^2 - R \cdot 2|\xi| \cdot t - |\xi|^2 t^2$$

$$\geq R^2 - 2|\xi| R - |\xi|^2$$

Hence, as $R\to\infty$, $Re((R-i\xi t)^2) \to \infty$ uniformly in $t\in[0,1]$.

$\therefore$ $\displaystyle\left| \int_{R}^{R-i\xi} e^{-\pi u^2}\, du \right| \to 0$ as $R\to\infty$.

#5 Assume f is continuously complex differentiable on Ω, and $T\subset\Omega$ is a triangle whose interior is contained in Ω. Apply Green's theorem to show

$$\int_T f(z)\, dz = 0.$$

Pf: Let $f(z) = u(x,y) + i v(x,y)$, where $z = x+iy$. Then

$$\partial_{\bar z} f = 0. \iff \begin{cases} \partial_x u = +\partial_y v \\ \partial_y u = -\partial_x v \end{cases}$$

$$\because \quad \partial_z f = \tfrac{1}{2}(\partial_x f - i\partial_y f)$$
$$= \tfrac{1}{2}(\partial_x(u+iv) - i\partial_y(u+iv))$$
$$= \tfrac{1}{2}(\partial_x u + \partial_y v) + i\tfrac{1}{2}(\partial_x v - \partial_y u)$$
$$= \partial_x u - i\partial_y u =$$

and $\partial_z f$ is a continuous function

$\therefore$ $\partial_x u$, $\partial_y u$ are continuous functions

and by CR equation, $\partial_x v$, $\partial_y v$ are continuous.

$$\int_T f\,dz = \int_T (u+iv)(dx+idy)$$

$$= \int_T (u\,dx - v\,dy) + i\int_T (v\,dx + u\,dy)$$

$$= \int_{int(T)} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\int_{int(T)} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$$
$$= 0.$$

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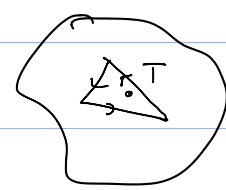
#6. Let f be a holomorphic function in $\Omega \setminus \{w\}$. and is bounded near w. Let T be a triangle in Ω containing w, then

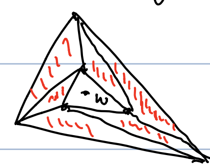
$$\int_T f(z)\,dz = 0$$



Pf: without loss of generality, we may assume $w=0$.

For $1>\varepsilon>0$, let εT denote the triangle T rescaled by ε.

Then, we may triangulate the polygon region between T and εT as shown.



By Goursat theorem, integral of $f(z)$ along any of the shaded triangles is zero. Hence.

$$\int_T f(z)\,dz = \int_{\varepsilon T} f(z)\cdot dz. \qquad \forall\ 1>\varepsilon>0.$$

However, since f is bounded on the solid triangle bounded by T,

$$\left|\int_T f(z)\,dz\right| = \left|\int_{\varepsilon T} f(z)\,dz\right| \leq C\cdot\varepsilon\cdot \text{length}(T) \qquad \forall\ \varepsilon>0$$

Thus. $\int_T f(z)\,dz = 0$.