Math 185: Homework 4 Solution

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The following exercises are from Stein's textbook, Chapter 2. 7,8,9,11,12

Problem (7)**.** *Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$ of the image satisfies*

$$2|f'(0)| \le d$$

Moreover it can be shown that equality holds precisely when f is linear, $f(z) = a_0 + a_1 z$.

Solution. For $0 < r < 1$, let C_r be the circle centered at 0 with radius r. Consider the Cauchy integral expression for $f'(0)$, we have

$$f'(0) = \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{(w-0)^2} dw.$$

We may replace the integration variable w by $-w$, and get

$$f'(0) = \frac{1}{2\pi i}\int_{C_r} \frac{f(-w)}{(-w-0)^2} d(-w).$$

Summing up the two equations, we have

$$2f'(0) = \frac{1}{2\pi i}\int_{C_r} \frac{f(w) - f(-w)}{w^2} dw.$$

Taking absolute value on both sides, we have

$$\begin{aligned} 2|f'(0)| &\le \frac{1}{2\pi}\int_{C_r} \frac{|f(w) - f(-w)|}{|w|^2} |dw| \\ &\le \frac{1}{2\pi} \sup_{w\in C_r} |f(w) - f(-w)| \cdot \int_{C_r} \frac{1}{|w|^2} |dw| \\ &\le \frac{1}{2\pi} d \frac{2\pi r}{r^2} \\ &= \frac{d}{r} \end{aligned}$$

Since the inequality holds for any $0 < r < 1$, we get

$$2|f'(0)| \le \inf_{0<r<1} \frac{d}{r} = d.$$

Remark. I wasn't able to figure out the 'more over' part. If one define $g(z) = \frac{f(z)-f(-z)}{d}$, then one get $g(0) = 0$, $|g'(0)| = 1$, and $(\mathbb{D}) \subset \mathbb{D}$, then by problem 9, one can show that $g(z) = z$. However, this only forces the odd part of $f(z)$ to be z, the information of the even part is lost when we construct $g(z)$. So, bounty for homework: the first one to completely solve the 'more over' part will get extra 5 points for the overall homework.

Problem (8). *If f is holomorphic on a strip $\{x + iy \mid x \in \mathbb{R}, -1 < y < 1\}$, with*

$$|f(z)| \leq A(1 + |z|)^\eta, \qquad \eta \text{ a fixed real number}$$

for all z in that strip. Show that for each $n \geq 0$, there exists a constant A_n, such that

$$|f^{(n)}(x)| \leq A_n(1 + |x|)^\eta$$

for all $x \in \mathbb{R}$.

Solution. Fix a r with $0 < r < 1$. Let $C_r(x)$ be the circle centered at x with radius r. Then by Cauchy estimate

$$|f^{(n)}(x)| \leq \frac{n!}{r^n} \sup_{w \in C_r(x)} |f(w)| \leq \frac{n!}{r^n} A \sup_{w \in C_r(x)} (1 + |w|)^\eta.$$

We claim that there exists a constant C, only dependent on η, such that

$$\sup_{w \in C_r(x)} (1 + |w|)^\eta < C(1 + |x|)^\eta$$

Given the claim, we have the desired result

$$|f^{(n)}(x)| \leq \inf_{0<r<1} \frac{n!}{r^n} AC(1 + |x|)^\eta = n!AC(1 + |x|)^\eta,$$

with $A_n = n!AC$.

Now we prove the claim. In fact we show one can take $C = 2^{|\eta|}$. Indeed, if $\eta > 0$, then

$$\sup_{w \in C_r(x)} (1 + |w|)^\eta \leq (1 + |x| + r)^\eta \leq (2 + |x|)^\eta = 2^\eta(1 + |x|/2)^\eta \leq 2^\eta(1 + |x|)^\eta.$$

If $\eta < 0$, then

$$\sup_{w \in C_r(x)} (1 + |w|)^\eta \leq \begin{cases} 1 & |x| < r \\ (1 + |x| - r)^\eta & |x| \geq r \end{cases} \leq \begin{cases} 1 & |x| < 1 \\ |x|^\eta & |x| \geq 1 \end{cases}$$

Let $h(x)$ be the piecewise defined function on the right in the above inequality. For $|x| < 1$,

$$\sup_{|x|<1} \frac{h(x)}{(1 + |x|)^\eta} = \sup_{|x|<1} \frac{1}{(1 + |x|)^\eta} = 2^{-\eta}$$

and for $|x| \geq 1$,

$$\sup_{|x|\geq 1} \frac{h(x)}{(1+|x|)^\eta} = \sup_{|x|\geq 1} \frac{|x|^\eta}{(1+|x|)^\eta} = 2^{-\eta}.$$

Hence for $\eta < 0$, we may take $C = 2^{-\eta}$, and get

$$\sup_{w\in C_r(x)} (1+|w|)^\eta < 2^{-\eta}(1+|x|)^\eta$$

. For $\eta = 0$, we can take $C = 1$.

Problem (9)**.** *Let Ω be a bounded open set of $\mathbb{C}$, and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a $z_0 \in \Omega$, such that*

$$\varphi(z_0) = z_0, \quad \varphi'(z_0) = 1$$

then φ is linear.

Solution. One can define $\Omega = \Omega - z_0$ and $\varphi(z) = \varphi(z_0+z) - z_0$, then $\varphi : \Omega \to \Omega$, and satisfies $\varphi(0) = 0, \varphi'(0) = 1$. $\varphi(z)$ is a linear function (i.e. of type $a + bz$) if and only if φ is a linear function. Hence without loss of generality, we may replace Ω by Ω, φ by φ and assume $z_0 = 0$.

Consider the power series expansion of φ,

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$$

which converges in a neighborhood of 0. By the assumption on φ, we have $a_0 = 0, a_1 = 1$. Assume there exists some other a_n non-zero, and let m be the smallest integer such that $m \geq 2$ and $a_m \neq 0$. Then, we have

$$\varphi(z) = z + a_m z^m + O(z^{m+1}).$$

Let $\varphi_k = \varphi \circ \cdots \circ \varphi$ denote the k-th iteration of φ. Then φ_k satisfies the same condition as φ, namely $\varphi_k(0) = 0, \varphi_k'(0) = 1$ (by chain rule). Furthermore, we claim that the Taylor expansion of $\varphi_k(z)$ at $z = 0$ is of the form

$$\varphi_k(z) = z + a_m k z^m + O(z^{m+1})$$

Indeed, one can prove this by induction on k. The case $k = 1$ is known. Suppose we have the expansion for index equals k, and we will prove it for index $k + 1$, then

$$\begin{aligned}
&\varphi_{k+1}(z) = \varphi(\varphi_k(z)) = \varphi_k(z) + a_m(\varphi_k(z))^m + O(\varphi_k(z)^m) \\
=&(z + a_m k z^m + O(z^{m+1})) + a_m(z + a_m k z^m + O(z^{m+1}))^m + O(z^{m+1}) \\
=&z + a_m k z^m + a_m z^m + O(z^{m+1}) \\
=&z + a_m(k+1)z^m + O(z^{m+1})
\end{aligned}$$

However this is in contradiction with the Cauchy estimate. Let $r > 0$ be chosen such that $\overline{D_r(0)} \subset \Omega$. Let $R = \sup_{z\in\Omega} |z|$. Then we have

$$m!|a_m|k = |\varphi_k^{(m)}(0)| \leq \frac{m!}{r^m} \sup_{z\in C_r(0)} |\varphi_k(z)| \leq \frac{m!}{r^m} R$$

The right hand side is independent of k, and as $k \to \infty$, the LHS is unbounded, hence there is a contradiction. Thus there is no $a_m \neq 0$ with $m \geq 2$.

Problem (11)**.** *Let f be a holomorphic function in the disk $D_{R_0}(0)$.*
(a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi}) Re\left(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}\right) d\varphi$$

(b) Show that

$$Re\left(\frac{Re^{i\varphi} + r}{Re^{i\varphi} - r}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\varphi + r^2}$$

Solution. Let $w = Re^{i\varphi}$. Then we try to write the integral as an integral for $w \in C_R(0)$ with integrand holomorphic in w. We have

$$f(Re^{i\varphi}) = f(w), \quad d\varphi = \frac{dw}{iw}$$

And

$$\mathrm{Re}\left(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}\right) = \mathrm{Re}\left(\frac{w+z}{w-z}\right) = \frac{1}{2}\left(\frac{w+z}{w-z} + \overline{\frac{w+z}{w-z}}\right) = \frac{1}{2}\left(\frac{w+z}{w-z} + \frac{\overline{w}+\overline{z}}{\overline{w}-\overline{z}}\right)$$

If $z = 0$, then $\mathrm{Re}(...) = 1$, and the result follows from Cauchy integral formula

$$f(0) = \frac{1}{2\pi i} \int_{C_R(0)} f(w) \frac{dw}{w}.$$

Now we assume $|z| > 0$. Note that on the circle $C_R(0)$, we have $w\overline{w} = |w|^2 = R^2$, hence we replace $\overline{w} = R^2/w$, and get

$$\frac{\overline{w}+\overline{z}}{\overline{w}-\overline{z}} = \frac{R^2/w + \overline{z}}{R^2/w - \overline{z}} = \frac{R^2/\overline{z} + w}{R^2/\overline{z} - w}.$$

The integral then become

$$\frac{1}{2\pi} \int_{C_R(0)} f(w) \frac{1}{2}\left(\frac{w+z}{w-z} + \frac{R^2/\overline{z} + w}{R^2/\overline{z} - w}\right) \frac{dw}{iw}$$

The integrand has three singularities, at $w = 0$, $w = z$ and $w = R^2/\overline{z}$. Note that $|R^2/\overline{z}| = R^2/|z| > R$. Hence only the pole 0 and z is inside the disk $D_R(0)$.

We may deform the contour $C_R(0)$ to two smaller circles $C_\epsilon(z)$ and $C_\epsilon(0)$ around the poles, then using Cauchy integral formula, we get

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{C_\epsilon(0)} f(w)\frac{1}{2}\left(\frac{w+z}{w-z}+\frac{R^2/\overline{z}+w}{R^2/\overline{z}-w}\right)\frac{1}{w}dw\\
=&\ f(w)\frac{1}{2}\left(\frac{w+z}{w-z}+\frac{R^2/\overline{z}+w}{R^2/\overline{z}-w}\right)\Big|_{w=0}\\
=&f(0)\frac{1}{2}(-1+1)=0,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{C_\epsilon(z)} f(w)\frac{1}{2}\left(\frac{w+z}{w-z}+\frac{R^2/\overline{z}+w}{R^2/\overline{z}-w}\right)\frac{1}{w}dw\\
=&\ f(w)\frac{1}{2}\left(\frac{w+z}{1}+\frac{R^2/\overline{z}+w}{R^2/\overline{z}-w}(w-z)\right)\frac{1}{w}\Big|_{w=z}\\
=&f(z)
\end{aligned}$$

Adding up the two terms, we get the desired equality.

(b)

$$\begin{aligned}
\mathrm{Re}\left(\frac{Re^{i\varphi}+r}{Re^{i\varphi}-r}\right)=\mathrm{Re}\frac{(Re^{i\varphi}+r)(Re^{-i\varphi}-r)}{(Re^{i\varphi}-r)(Re^{-i\varphi}-r)}&=\frac{\mathrm{Re}(R^2-r^2+rR(e^{i\varphi}-e^{-i\varphi}))}{R^2+r^2+2Rr\cos\varphi}\\
&=\frac{R^2-r^2}{R^2+r^2+2Rr\cos\varphi}
\end{aligned}$$

Problem (12)**.** *Let u be a real valued function defined on the unit disk $\mathbb{D}$. Suppose that u is twice differentiable and harmonic, that is $\Delta u(x,y)=0$ for all $x,y\in\mathbb{D}$.*

(a) Prove that there exists a holomorphic function f on the unit disk, such that

$$Re(f)=u$$

(b) Deduce from this result, the Poisson integration formula. If u is harmonic in $\mathbb{D}$ is is continuous on its closure $\overline{\mathbb{D}}$, then if $z=re^{i\theta}$, one has

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}P_r(\theta-\varphi)u(e^{i\varphi})d\varphi$$

where $P_r(\theta)$ is the Poisson kernel

$$P_r(\theta)=\frac{1-r^2}{1-2r\cos\theta+r^2}$$

Solution. (a) Let's make some observation first. To construct f, we try to construct its derivative f' then integrate to get f. If we know a holomorphic function $f(z)=u(z)+iv(z)$, then $f'(z)=2\partial_z u(z)$, indeed

$$\partial_z f(z)=\partial_x f(x,y)=\partial_x(u(x,y)+iv(x,y))=\partial_x u-i\partial_y u=2\partial_z u(z).$$

Now we begin the proof. Define $g(z) = 2\partial_z u(z) = \partial_x u - i\partial_y u$. Then $g(z)$ is once differentiable (though $\partial_x g, \partial_y g$ may not be continuous), since

$$\partial_{\bar{z}} g(z) = 2\partial_{\bar{z}}\partial_z u(z) = (1/2)\Delta u = 0$$

hence $g(z)$ is holomorphic for all $z \in \mathbb{D}$. From Theorem 2.1, we know g has a primitive F. We claim that $\mathrm{Re}F - u$ is a constant. Indeed, we have

$$\partial_x(\mathrm{Re}F - u) + i\partial_y(\mathrm{Re}F - u) = 2\partial_z(\mathrm{Re}F - u) = g(z) - g(z) = 0,$$

hence the partial derivatives of $(\mathrm{Re}F - u)$ vanishes, hence $\mathrm{Re}F - u$ is a constant. Denote this constant by c, and define $f = F - c$, we then get f a holomorphic function with $\mathrm{Re}f = u$.

(b) Apply (a) to get a holomorphic function f with $\mathrm{Re}f = u$. If $z = e^{i\theta}$, then let $R \in \mathbb{R}$ such that $|z| < R < 1$. Then by Exercise 11, we have

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\mathrm{Re}\left(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}\right) d\varphi$$

Taking the real part on both sides, we get

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\varphi})\frac{R^2 - r^2}{R^2 - 2Rr\cos(\varphi - \theta) + r^2} d\varphi.$$

Let $R \to 1$, by uniform continuity, we get

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})P_r(\theta - \varphi)d\varphi.$$

Note that $\cos(x)$ is an even function, hence $P_r(\theta - \varphi) = P_r(\varphi - \theta)$.