Stein, Ch3. Ex. 1.2.3.7.

1. Using Euler's formula

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i},$$

show that the complex zeros of $\sin \pi z$ are exactly at the integers, and that they are each of order 1.

Calculate the residue of $1/\sin \pi z$ at $z = n \in \mathbb{Z}$.

- If $z = n$ is an integer, it's easy to check that $\sin(n\pi) = 0$.

Conversely, if $\sin(z) = 0$, then $e^{\pi i z} = e^{-\pi i z}$

$\Leftrightarrow \quad e^{2\pi i z} = 1$

which is only possible if $2\pi i z = 2\pi i \cdot n$ for $n \in \mathbb{Z}$.

- To check the order of the zero at $z = n$ is 1, suffice to check that $\sin(\pi z)'\big|_{z=n} \neq 0$.

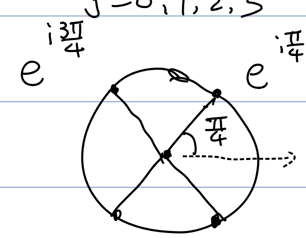
$$\sin(\pi z)'\Big|_{z=n} = \pi \cos(\pi z)\big|_{z=n} = \pi(-1)^n \neq 0. \qquad \#.$$

- $$\operatorname{Res}_{z=n} \frac{1}{\sin(\pi z)} = \frac{1}{\sin(\pi z)'\big|_{z=n}} = (-1)^n \frac{1}{\pi}$$

#2 Evaluate $\int_{-\infty}^{+\infty} \frac{1}{1+x^4}\, dx$.

The pole of $1/1+z^4$ is at the zeros of z^4+1,

$z^4+1=0 \Leftrightarrow z^4 = -1 \Leftrightarrow z = e^{\frac{2\pi i}{4} \cdot j + \frac{\pi i}{4}}$ $j = 0, 1, 2, 3$



$$\operatorname{Res}_{z=e^{i\frac{\pi}{4}}} \frac{1}{1+z^4} = \lim_{z\to e^{i\frac{\pi}{4}}} \frac{z - e^{i\frac{\pi}{4}}}{z^4+1} = \frac{1}{4z^3}\Big|_{z=e^{i\frac{\pi}{4}}}$$

$$= \frac{z}{4z^4}\Big|_{z=e^{i\frac{\pi}{4}}} = -\frac{1}{4} e^{i\frac{\pi}{4}}$$

$$\mathrm{Res}_{z=e^{i\frac{3\pi}{4}}} \frac{1}{1+z^4} = -\frac{1}{4} z\Big|_{z=e^{i\frac{3\pi}{4}}} = -\frac{1}{4} \cdot e^{i\frac{3\pi}{4}}$$

$$\therefore\ I = 2\pi i \cdot \sum \mathrm{Res}_z f = 2\pi i \left(-\frac{1}{4} \cdot e^{i\frac{\pi}{4}} + \left(-\frac{1}{4}\right) e^{i\frac{3\pi}{4}} \right)$$

Residue in the upper half plane

$$= 2\pi i \cdot \left(-\frac{1}{4}\right)\left(\sqrt{2}\, i\right)$$

$$= \frac{\sqrt{2}}{2}\pi$$

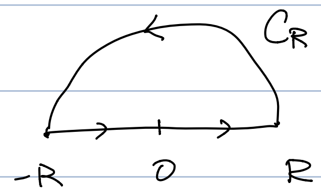
same argument as in the book.

#3 Show that $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+a^2}\, dx = \pi \cdot \frac{e^{-a}}{a}$ for $a>0$.

Let I denote the integral, then

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos x}{x^2+a^2}\, dx = \mathrm{Re}\left\{ \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{x^2+a^2}\, dx \right\}.$$

Complete the contour by adding an upper semi-circle and denote the total contour as C, $C = [-R, R] + C_R$



$$\int_C \frac{e^{iz}}{z^2+a^2}\, dz = \left(\mathrm{Res}_{z=ia} \frac{e^{iz}}{z^2+a^2} \right) \cdot 2\pi i$$

$$= \frac{e^{-a}}{2ia} \cdot 2\pi i$$

$$= \pi \frac{e^{-a}}{a}$$

$$\left| \int_{C_R} \frac{e^{iz}}{z^2+a^2}\, dz \right| \leq \frac{1}{R^2+a^2} \cdot 2\pi R \longrightarrow 0 \quad \text{as } R\to\infty$$

Hence $\displaystyle\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{x^2+a^2}\, dx = \oint \frac{e^{iz}}{z^2+a^2}\, dz - \lim_{R\to\infty} \int \frac{e^{iz}}{z^2+a^2}\, dz$

$$= \pi \frac{e^{-a}}{a}.$$ Note that it is already a real number.

Hence $I = \pi \cdot \frac{e^{-a}}{a}.$

#7. 7. Prove that

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}}, \qquad \text{whenever } a > 1.$$

Let $e^{i\theta} = z$, then $d(e^{i\theta}) = dz$

$\Leftrightarrow \quad e^{i\theta} \cdot i \cdot d\theta = dz$

$\Leftrightarrow \quad d\theta = \frac{1}{iz} dz$

And. $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + 1/z}{2}$

$$I = \oint_{|z|=1} \frac{1}{\left(a + \frac{z+1/z}{2}\right)^2} \cdot \frac{dz}{iz} = \oint_{|z|=1} \frac{(2z)^2}{(2az + z^2+1)^2} \cdot \frac{dz}{iz}$$

Next, we find root for the denominator

$z^2 + 2az + 1 = 0 \quad \Leftrightarrow \quad (z+a)^2 - a^2 + 1 = 0$

$\Leftrightarrow \quad (z+a)^2 = a^2 - 1$

$\Leftrightarrow \quad z + a = \pm\sqrt{a^2-1}$

$\Leftrightarrow \quad z = -a \pm \sqrt{a^2-1}$

Let $z_\pm = -a \pm \sqrt{a^2-1}$, then $|z_+| < 1$, $|z_-| > 1$.

Then we have an order 2 pole at $z = z_+$ within $|z| < 1$.

$$\operatorname{Res}_{z=z_+} \frac{(2z)^2/iz}{(z-z_+)^2(z-z_-)^2} = \lim_{z\to z_+} \left(\frac{-4iz}{(z-z_-)^2} \right)'$$

$$= \frac{-4i}{(z-z_-)^2} + \frac{(-4iz)(-2)}{(z-z_-)^3}\Bigg|_{z=z_+} = \frac{(-4i)(z_+ - z_- - 2z_+)}{(z_+ - z_-)^3}$$

$$= \frac{(-4i)\cdot(+2a)}{(2\sqrt{a^2-1})^3} = \frac{-ia}{(a^2-1)^{3/2}}$$

$$\therefore\ I = 2\pi i \cdot \operatorname{Res}_{z=z_+}(\text{integrand}) = 2\pi i \cdot \frac{-ia}{(a^2-1)^{3/2}} = \frac{2\pi a}{(a^2-1)^{3/2}}$$

(ok. this is a messy problem).

#5: Let $P(z) = a_n \cdot z^n + \cdots + a_0$

$$= a_n(z-z_1)(z-z_2)\cdots(z-z_n)$$

for $z_1, \cdots, z_n$ in $\mathbb{D}$, by assumption.

Then $$I = \oint_{\substack{|z|=1 \\ \text{CCW}}} \frac{1}{P(z)}\,dz = \oint_{\substack{|w|=1 \\ \text{CW}}} \frac{1}{P(\frac{1}{w})}\,d\left(\frac{1}{w}\right) = -\oint_{\substack{|w|=1 \\ \text{CCW}}} \frac{w^n \left(-\frac{1}{w^2}\right)dw}{a_n(1-z_1w)\cdots(1-z_nw)}$$

CCW || counterclockwise

CW clockwise

Claim: $\tilde{P}(w) = (1-z_1w)\cdots(1-z_nw)$

is a polynomial of degree $\le n$, with no roots in $|w|<1$.

Pf: if z_i are all distinct and none zero, then

$$\tilde{P}(w) = z_1\cdots z_n(-1)^n(w - 1/z_1)\cdots(w-1/z_n)$$

a degree n polynomial, with roots $\frac{1}{z_1}, \cdots, \frac{1}{z_n}$, all outside of $|w|<1$. If some z_i are zero, say $z_{m+1}, \cdots, z_n$ are zero, then

$\tilde{P}(w) = (1 - z_1 w) \cdots (1 - z_m \cdot w)$, hence is a degree m polynomial. And the rest is the same as before.

$$I = \oint_{|w|=1} \frac{w^{n-2}}{a_n \tilde{P}(w)} \, dw = 0$$

since $\frac{w^{n-2}}{\tilde{P}(w)}$ is a holomorphic function inside $|w| \le 1$.

Extra version: $\deg Q \le \deg P - 2$. say $\deg P = n$, $\deg Q = m$

$$I = \oint_{|z|=1} \frac{Q(z)}{P(z)} \, dz = + \oint_{|w|=1} \frac{Q(\frac{1}{w})}{P(\frac{1}{w})} \frac{1}{w^2} \, dw = \oint \frac{w^{n-2} Q(\frac{1}{w})}{\tilde{P}(w)} dw$$

Then since $w^{n-2} Q(\frac{1}{w}) = w^{n-2-m} \left(w^m \, Q(\frac{1}{w}) \right)$ is an entire function, and $\tilde{P}(w)$ has no root inside $|w| \le 1$, hence $I = 0$.