Stein 9. 10. 14. 15. 16 in Ch 3.

#9. Show that $\int_0^1 \log(\sin \pi x)\,dx = -\log 2$.

<u>Sol'n:</u> it is equivalent to show

$$\int_0^{\pi} \log(\sin x)\,dx = -\pi \log 2.$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{iz}}{2i}\left(1 - e^{-2iz}\right).$$

To fix the branch of log, we choose the anchor point $z_0 = \frac{\pi}{2}$.
Here $\sin z_0 = 1$, indeed.

$$\frac{e^{i\frac{\pi}{2}}}{2i}\left(1 - e^{-2i\cdot\frac{\pi}{2}}\right) = \frac{i}{2i}\left(1-(-1)\right) = 1.$$

Now, apply log, we have

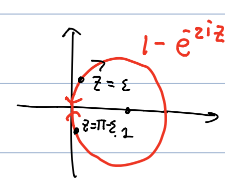
$$\begin{aligned}
\log(\sin z) &= \log\left(\frac{1}{2i}\right) + \log\left(e^{iz}\right) + \log\left(1 - e^{-2iz}\right). \\
&= -\log\left(2e^{i\frac{\pi}{2}}\right) + (iz) + \log\left(1 - e^{-2iz}\right). \\
&= -\log 2 - i\frac{\pi}{2} + iz + \log\left(1 - e^{-2iz}\right).
\end{aligned}$$

Indeed, for $z = z_0 = \frac{\pi}{2}$, the above expression is $\log(1) = 0$.

$\because \int_0^{\pi} -i\frac{\pi}{2} + iz\,dz = 0$, $\int_0^{\pi} -\log 2\,dz = -\pi \log 2$.

$\therefore$ Our main task is then to prove that

$$\int_0^{\pi} \log\left(1 - e^{-2iz}\right) dz = 0$$

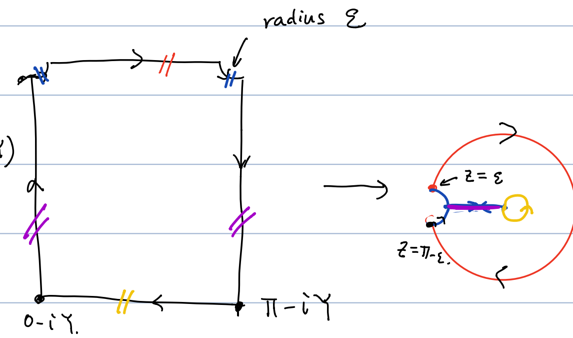


Since as $z \to 0$ or $z \to \pi$. $\log\left(1 - e^{-2iz}\right) \to -\infty$,

we should ~~comp.~~ prove

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\pi-\varepsilon} \log(1-e^{-2iz})\, dz = 0.$$

Consider the following contour



for $z = x+iy$ $\quad x \in (0,\pi),\ y \in (0,-Y)$

$$1-e^{-2iz} = 1-e^{-2ix+2y}$$

$$= 1-e^{2y}(\cos 2x - i\sin 2x)$$

Hence in the interior and boundary of the enclosed region. $1-e^{-2iz}$ has real part positive., and $\log(1-e^{-2iz})$ is single valued and holomorphic. Thus.

In the various segments that make up the contour, the yellow part integrate to 0;

$$\int_0^{\pi} \log(1-e^{-2Y}e^{-2i\theta})\, d\theta$$

$$= \int_0^{2\pi} \log(1-e^{-2Y}e^{+i\theta})\, d\theta$$

$$= 0 \qquad \text{by mean value thm to } \log(1-e^{-2Y}z).$$

(vertical two lines).

The purple parts cancels out.

The small radius ε arc: the arc near $z=0$. is.

$$\int_{\substack{z=\varepsilon\cdot e^{i\theta} \\ \theta\in(-\frac{\pi}{2},0)}} \log(1-e^{-2iz})\, dz \longrightarrow 0 \quad \text{as } \varepsilon\to 0$$

Indeed, the imaginary part of $\log(1-e^{-2iz})$ is bounded, hence we only need to consider the real part

Since $|1-e^{-2iz}|/|z| \to 2$ as $|z| \to 0$, we have.

$$\left| \int_{\text{small arc}} \log|1-e^{-2iz}| \cdot dz \right| \le C \cdot \varepsilon \cdot \log \varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Similar argument for the small arc near $z = \pi$.

Thus, we have concluded that

$$\lim_{\varepsilon \to 0} \left(\int_{\varepsilon}^{\pi-\varepsilon} \log(1-e^{-2iz})\, dz + \int_{\substack{\text{small} \\ \text{arcs}}} \log(1-e^{-2iz})\, dz \right.$$

$$\left. + \text{purple contour} + \text{yellow contour} \right) = 0$$

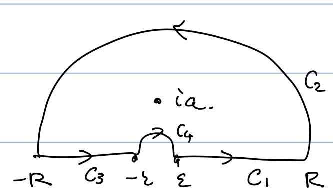
and since the last 3 terms are zero (after $\varepsilon \to 0$ limit), hence the first term is also zero.

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\pi-\varepsilon} \log(1-e^{-2iz})\, dz = 0. \qquad \#$$

10 Show that if $a > 0$, then

$$\int_0^\infty \frac{\log x}{x^2+a^2}\, dx = \frac{\pi}{2a} \log a.$$

$$\text{LHS} = \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_{\varepsilon}^{R} \frac{\log x}{x^2+a^2}\, dx.$$



Consider the following contour

$C = C_1 + C_2 + C_3 + C_4$. $\log z$ in the region enclosed by C is single valued and holo.

$$\int_C \frac{\log z}{z^2+a^2}\, dz = 2\pi i \operatorname{Res}_{z=ia} \frac{\log z}{z^2+a^2} = \frac{\log(ia)}{2ia} 2\pi i$$

Denote $I_i = \int_{C_i} \frac{\log z}{z^2+a^2}\, dz$, then.

$$I_3 = \int_{C_3} \frac{\log(z)}{z^2+a^2}\, dz = \int_{\varepsilon}^{R} \frac{\log(\rho\cdot e^{\pi i})}{\rho^2+a^2}\, d\rho$$

$$= \int_{\varepsilon}^{R} \frac{\log \rho}{\rho^2+a^2} + \frac{\pi i}{\rho^2+a^2}\, d\rho = I_1 + (\pi i)\int_{\varepsilon}^{R} \frac{1}{\rho^2+a^2}\, d\rho$$

$$I_4 = \int_{\pi}^{0} \frac{\log(\varepsilon\cdot e^{i\theta})}{\varepsilon^2 e^{2i\theta}+a^2}\, d(\varepsilon\cdot e^{i\theta}) .$$

$$= \varepsilon\cdot \int_{\pi}^{0} \frac{\log\varepsilon + i\theta}{\varepsilon^2 e^{2i\theta}+a^2}\, e^{i\theta}\cdot i\cdot d\theta$$

$$|I_4| \leq (C_1 + C_2 \log\varepsilon)\,\varepsilon \to 0 \quad \text{as } \varepsilon \to 0.$$

$$I_2 = \int_{0}^{\pi} \frac{\log(R\cdot e^{i\theta})}{(Re^{i\theta})^2+a^2.}\, Re^{i\theta}\, i\, d\theta$$

$$|I_2| \leq \frac{1}{R}\,(C_1 + C_2 \log R) \to 0 \quad \text{as } R \to \infty.$$

Finally. $$\int_{\varepsilon}^{R} \frac{1}{\rho^2+a^2}\, d\rho \xrightarrow[R\to\infty]{\varepsilon\to 0} \int_{0}^{\infty} \frac{1}{\rho^2+a^2}\, d\rho = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{\rho^2+a^2}\, d\rho$$

$$= \frac{1}{2}\cdot 2\pi i\cdot \frac{1}{2ia.}$$

Hence $$2\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} I_1 + \frac{1}{2}\, 2\pi i\cdot \frac{\pi i}{2ia} = 2\pi i\cdot \frac{\log a + i\frac{\pi}{2}.}{2ia.}$$

$$\therefore\ 2\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} I_1 = 2\pi i\cdot \frac{\log a}{2ia.}$$

$$\text{LHS} = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} I_1 = \frac{\pi\log a}{2a.}$$

#14 Prove that entire function that are also injective take the form $f(z) = az + b$.

Pf: consider $g(w) = f(\frac{1}{w})$, and study the possible singularities at $w=0$.

- $w = 0$ is a removable singularity, then $f(z)$ is hol'c on $\hat{\mathbb{C}}$, in particular boundal at $z=\infty$. Hence f = constant by Liouville thm. Contradicting with injectivity.
- $w=0$ is an essential singularity. Then by Casorati-Weierstrass thm., for any $R>0$.

$g(\{w \mid 0<|w|<\frac{1}{R}\}) = f(\{|z|>R\})$ is dense in $\mathbb{C}$.

by open mapping thm $f(\{|z|<R\})$ is open in $\mathbb{C}$.

Hence

$$f(\{|z|<R\}) \cap f(\{|z|>R\}) \neq \phi.$$

Contradicting f is injective. Hence $w=0$ is not an essential singularity.

- $w=0$ is a pole. Assume the pole is order n ($n \geq 1$). then $f(z)$ is a degree n polynomial. By injectivity, $n=1$. Hence, we can write

$f(z) = az+b$ for some $a, b \in \mathbb{C}$, $a \neq 0$.

#15 (a) If f is an entire function satifies.

$$\sup_{|z|=R} |f(z)| \leq A \cdot R^k + B .$$

then f is a polynomial of deg $\leq k$.

Pf: Consider $F(z) = \frac{f(z)}{z^k}$, then $F(z)$ at $z=0$ has a pole at most order k.

$$\sup_{|z|=R} |F(z)| \leq A + \frac{B}{R^k}.$$

and $F(z)$ as $z \to \infty$ is bounded, hence ∞ a removable singularity. We may write

$F\left(\frac{1}{w}\right)$ as a polynomial in w of degree at most k.

$$F\left(\frac{1}{w}\right) = a_0 + \cdots + a_k \cdot w^k \qquad \text{for } a_i \in \mathbb{C}.$$

thus

$$f(z) = F(z) \cdot z^k = z^k\left(a_0 + a_1 \frac{1}{z} + \cdots + a_k \frac{1}{z^k}\right)$$

$$= a_0 \cdot z^k + a_1 z^{k-1} + \cdots + a_k.$$

#

(b) Show that, if f is holomophic in the unit disk, bounded, and converges uniformly to zero on sector $\theta < \arg z < \varphi$ as $|z| \to 1$.

Then $f = 0$.

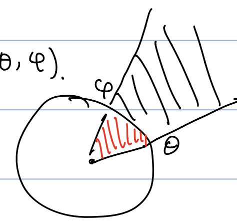
Pf: For $|z|<1$, define $z^* = \frac{1}{\bar{z}}$, then

$\arg(z^*) = \arg(z)$ and $|z^*| \cdot |z| = 1$.

we apply Schwarz reflection principle along the arc $\{ e^{i\phi} \mid \theta < \phi < \varphi \}$, and define for $r>1$

$$f(r \cdot e^{i\phi}) = \overline{f(\tfrac{1}{r} e^{i\phi})}$$

or. $f(z^*) = \overline{f(z)}$ for $0<|z|<1$, $\arg(z) \in (\theta, \varphi)$.



(*) One may check that f is hol'c.

for $0<|z|<\infty$ $\arg(z) \in (\theta, \varphi)$.

and $f(z) = 0$ for $|z|=1$, $\arg(z) \in (\theta, \varphi)$.

This forces $f \equiv 0$ on its domain.

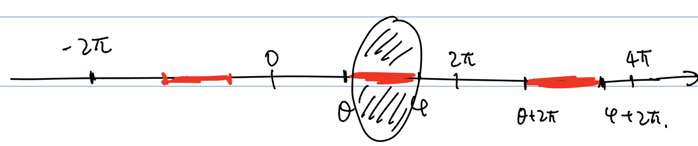
To check (*):

We may define a function $F(\omega)$ for $\mathrm{Im}(\omega) > 0$,

$$F(\omega) = f(e^{i\omega}) \qquad \because \mathrm{Im}(\omega) > 0, \quad \therefore |e^{i\omega}| < 1.$$

then the segment $\arg(e^{i\omega}) \in (\theta, \varphi)$

$\iff \omega \in (\theta, \varphi) + 2\pi n$ for some $n \in \mathbb{Z}$.



Hence, we can extend F across these segments $(\theta, \varphi) + 2\pi\mathbb{Z}$.

by usual Schwarz reflection principle.

(c) Let $w_1, \cdots, w_n$ be points on the unit circle. C

Prove that there exists a point $z \in C$, such that

$$|(z-w_1)\cdots(z-w_n)| \geq 1.$$

and furthermore, prove that z can be chosen, such that

$$|(z-w_1)\cdots(z-w_n)| = 1.$$

Pf: let $f(z) = (z-w_1)\cdots(z-w_n)$.

Then $|f(0)| = |w_1|\cdots|w_n| = 1$. By Maximum principle,

$$\sup_{|z|\leq 1} |f(z)| = \sup_{|z|=1} |f(z)| \geq |f(0)| = 1.$$ Hence the first claim.

As z move towards the nearest w_j on the unit circle, $|f(z)|$ goes from ≥ 1 to $\to 0$, hence at certain place, $|f(z)| = 1$.

(d) If the real part of an entire function f is bounded, then f is constant.

Pf: Consider $e^{f(z)}$, then $|e^f| = e^{\operatorname{Re} f}$ is Bounded

hence by Liouville thm is constant.

Thus $(e^f)' = f' \cdot e^f = 0 \Rightarrow f' = 0$

$\Rightarrow f$ is constant.

#16

16. Suppose f and g are holomorphic in a region containing the disc $|z| \le 1$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $|z| \le 1$. Let

$$f_\epsilon(z) = f(z) + \epsilon g(z).$$

Show that if ϵ is sufficiently small, then

(a) $f_\epsilon(z)$ has a unique zero in $|z| \le 1$, and

(b) if z_ϵ is this zero, the mapping $\epsilon \mapsto z_\epsilon$ is continuous.

pf (a) follows from Rouché thm

(b). Suppose $\varepsilon_0 > 0$ is small enough, such that

$\forall \varepsilon \in \mathbb{C}$, $0 \le |\varepsilon| < \varepsilon_0$, $f_\varepsilon(z)$ has unique zero z_ε inside $|z| \le 1$.

and $|z_\varepsilon| < 1$. Then.,

$$z_\varepsilon = \frac{1}{2\pi i} \int_C \frac{f_\varepsilon'(z)}{f_\varepsilon(z)} z \, dz$$

indeed. $\dfrac{f_\varepsilon'(z)}{f_\varepsilon(z)} = \left(\dfrac{1}{z - z_\varepsilon} + \text{hol'c part} \right)$.

Hence $\dfrac{1}{2\pi i} \displaystyle\int_C \left(\frac{1}{z - z_\varepsilon} + \text{hol'c} \right) \cdot z \, dz = z_\varepsilon$.

Since the integrand $\dfrac{f_\varepsilon'(z)}{f_\varepsilon(z)} \cdot z$ is continuous in ε., for $|\varepsilon| < \varepsilon_0$, and domain C is compact, hence the integrand is uniformly continuous in ε. Thus the result of the integral is continuous in ε.

actually, the integrand is hol'c in ε. and z_ε is hol'c in ε for $|\varepsilon| < \varepsilon_0$

Or, one can prove continuity by implicit function thm.