1. Taylor expand $\log\left(\frac{\sin z}{z}\right)$ around $z=0$. upto $O(z^6)$.
(not including z^6 term).

sol'n:
$$\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \cdots$$
$$\frac{\sin z}{z} = 1 \underbrace{- \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \frac{1}{7!}z^6 + \cdots}_{\alpha}$$
$$= 1 + \alpha,$$

For clarity, we keep the z^6 term until the very end.
$$\log(1+\alpha) = \alpha - \frac{1}{2}\alpha^2 + \frac{1}{3}\alpha^3 - \cdots$$
$$= \left(-\frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \frac{1}{7!}z^6 + \cdots\right) - \frac{1}{2}\left(-\frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \cdots\right)^2$$
$$+ \frac{1}{3}\left(-\frac{1}{3!}z^2 + \cdots\right)^3$$
$$= \left(-\frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \cdots\right) - \frac{1}{2}\left(-\frac{1}{3!}z^2\right)^2 + \cdots$$
$$= -\frac{1}{6}z^2 + \frac{1}{120}z^4 - \frac{1}{2}\frac{1}{36}\cdot z^4 + \cdots$$
$$= -\frac{1}{6}z^2 - \frac{1}{180}z^4 + \cdots$$

2. Let f be a hol'c function on the annulus $R_1 < |z| < R_2$.
Prove that the Laurent series expansion of f is unique.

Pf: If $f(z) = \sum_{n=-\infty}^{+\infty} a_n \cdot z^n$ for $R_1 < |z| < R_2$,

Then for any $r \in \mathbb{R}$, with $R_1 < r < R_2$, we have
$\sum_{n=-\infty}^{\infty} |a_n| \cdot r^n < \infty$.

Hence, we may extract a_n using Cauchy integral formula.
$$\frac{1}{2\pi i}\int_{|z|=r} f(z)\cdot z^{-k-1}\, dz = \frac{1}{2\pi i}\int_{|z|=r} \sum_{n\in\mathbb{Z}} a_n \cdot z^{n-k-1}\, dz$$
$$= \frac{1}{2\pi i}\sum_{n\in\mathbb{Z}}\int a_n \cdot z^{n-k-1}\, dz = a_k$$

$|z|=r$

where the change of order of summation and integral is justified, since if we let

$$F_N(z) = \sum_{n=-N}^{N} a_n \cdot z^{n-k-1}, \quad F(z) = \lim_{N\to\infty} F_N(z).$$

then $\{F_N(z)\}$ converge to $F(z)$ uniformly on $\{|z|=r\}$.

Thus, the coefficient is determined by an integral, hence is unique.

3. Express $\sum_{n=-\infty}^{+\infty} \frac{1}{z^3-n^3}$ in closed form.

Sol'n: Let $\omega = e^{\frac{2\pi i}{3}}$ be a cubic root of unity.

Then

$$z^3-n^3 = (z-n)(z-\omega\cdot n)(z-\omega^2 n).$$

If $n\neq 0$, we have.

$$\frac{1}{z^3-n^3} = \frac{1}{z-n}\cdot\frac{1}{(n-\omega n)(n-\omega^2 n)} + \frac{1}{(z-\omega n)(\omega n-n)(\omega n-\omega^2 n)}$$

$$+ \frac{1}{(z-\omega^2 n)}\,\frac{1}{(\omega^2 n-n)(\omega^2 n-\omega n)}$$

$$= \frac{C}{n^2}\left(\frac{1}{z-n} + \frac{1}{\omega z-n} + \frac{1}{\omega^2 z-n}\right)$$

where constant $C = \frac{1}{(1-\omega)(1-\omega^2)} = \frac{1}{1-\omega-\omega^2+1} = \frac{1}{3}$.

Consider the first term., $\sum_{n\neq 0} \frac{1}{n^2}\frac{1}{z-n}$

(notice, this sum is absolutely convergent, thanks to the extra $\frac{1}{n^2}$ factor). We want compare $\frac{1}{n^2}\frac{1}{z-n}$ with $\frac{1}{z^2}\frac{1}{z-n}$, since they have the same location of the pole and same residue. We define.

$$F_N(z) = \sum_{n=1}^{N} \frac{1}{z^2}\left(\frac{1}{z-n} + \frac{1}{z+n}\right)$$

$$G_N(z) = \sum_{n=1}^{N} \frac{1}{n^2}\left(\frac{1}{z-n} + \frac{1}{z+n}\right).$$

then $$F_N(z) - G_N(z) = \sum_{n=1}^{N}\left(\frac{1}{z^2} - \frac{1}{n^2}\right)\left(\frac{2z}{z^2-n^2}\right)$$

$$= \sum_{n=1}^{N} \frac{n^2-z^2}{z^2 n^2}\cdot\frac{2z}{z^2-n^2} = \frac{-2}{z}\sum_{n=1}^{N}\frac{1}{n^2}$$

And note. $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

Using $\pi\cot(\pi z) = \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{z-n}$, we have.

$$\lim_{N\to\infty} F_N(z) = \frac{1}{z^2}\left(\pi\cot(\pi z) - \frac{1}{z}\right)$$

Thus. $$\lim_{N\to\infty} G_N(z) = \lim_{N\to\infty} F_N(z) - \lim_{N\to\infty}(F_N - G_N)(z)$$

$$= \frac{1}{z^2}\left(\pi\cot\pi z - \frac{1}{z}\right) + \frac{2}{z}\cdot\frac{\pi^2}{6}$$

$$=: G(z).$$

$$\therefore \sum_{n=-\infty}^{+\infty}\frac{1}{z^3-n^3} = \frac{1}{z^3} + \sum_{n\neq 0}\frac{1/3}{n^2}\left(\frac{1}{z-n} + \frac{1}{z\omega-n} + \frac{1}{z\cdot\omega^2-n}\right)$$

$$= \frac{1}{z^3} + \frac{1}{3}\left(G(z) + G(\omega z) + G(\omega^2 z)\right)$$

$$= \frac{1}{3}\left(\frac{\pi\cot(\pi z)}{z^2} + \frac{\pi\cot(\pi\cdot\omega z)}{\omega^2\cdot z^2} + \frac{\pi\cot(\pi\cdot\omega^2 z)}{\omega\cdot z^2} \right)$$

where $\frac{1}{z^3}$ term cancels out, and $\frac{1}{z}$ linear term has a factor $(1+\omega+\omega^2)=0$, also drops out.

#4 Prove that

$$\prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)\cdot e^{-z/n}$$

converges absolutely and uniformly on every compact subset.

Pf: · Note that there exists $C>0$, $\varepsilon>0$, such that $\forall\, |\alpha|<\varepsilon$, we have

$$|\log(1+\alpha)-\alpha| \le C\cdot \alpha^2.$$

· Fix any $R>0$. We may choose N large enough (integer), such that $\left|\frac{R}{N}\right| < \varepsilon$. Then, $\forall\, |z|\le R$,

$$\sum_{n=N}^{\infty} \left|\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right| \le \sum_{n=N}^{\infty} C\cdot\frac{|z|^2}{n^2} < \infty$$

Hence $\prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)\cdot e^{-\frac{z}{n}}$ converges absolutely and uniformly on $\{|z|\le R\}$.

#5. Show that, the function

$$\theta(z) = \prod_{n=1}^{\infty}(1+h^{2n-1}\cdot e^{z})(1+h^{2n-1}e^{-z})$$

where $|h|<1$ is analytic in the whole plane, and

$$\theta(z+2\log h) = h^{-1}\cdot e^{-z}\cdot\theta(z).$$

Pf: To show $\theta(z)$ is analytic, suffice to show that the two products $\prod_{n=1}^{\infty}(1+h^{2n-1}\cdot e^{z})$ and $\prod_{n=1}^{\infty}(1+h^{2n-1}\cdot e^{-z})$ converges absolutely and uniformly on compact subsets in $\mathbb{C}$.

Indeed, for any compact subset $E\subset\mathbb{C}$, there is a constant M, s.t. $M>|e^{z}|$ and $M>|e^{-z}|$, $\forall z\in E$.

Then $\because \sum_{n=1}^{\infty}|h^{2n-1}\cdot e^{z}| = |e^{z}|\cdot|h|/(1-|h|^2)<\infty$

Hence the product converges absolutely and uniformly $\forall z\in E$.

Similarly for the other product with e^{-z} replacing e^{z}.

$$\theta(z+2\log h) = \prod_{n=1}^{\infty}(1+h^{2n-1}\cdot e^{z+2\log h})\cdot\prod_{n=1}^{\infty}(1+h^{2n-1}e^{-z-2\log h})$$

$$= \prod_{n=1}^{\infty}(1+h^{2n+1}\cdot e^{z})\cdot\prod_{n=1}^{\infty}(1+h^{2n-3}\cdot e^{-z})$$

$$\overset{(*)}{=} \theta(z)\cdot\frac{1+h^{-1}e^{-z}}{1+h\cdot e^{z}} = \theta(z)\frac{h^{-1}e^{-z}(1+h\cdot e^{z})}{1+h\cdot e^{z}}$$

$$= \theta(z)\,h^{-1}\cdot e^{-z}$$

- Note, the equality $(*)$ holds if $1+h\cdot e^{z}\neq 0$. If $1+h\cdot e^{z}=0$, then both sides vanishes as well, hence equality also holds.

- Note., $\log h$ has ambiguity of $2\pi i \cdot \mathbb{Z}$. However, if we change z by $2\pi i \cdot \mathbb{Z}$, e^{z} and e^{-z} is invariant with this shift.