Today : • more on examples of contour integral
• derivatives of Cauchy integral formula.

1. $$\int_0^\infty \frac{1-\cos x}{x^2}\, dx = \frac{\pi}{2}$$

∘ Why this integral is convergent at $x \to 0$ and $x \to \infty$?

∘ near 0 : $1-\cos x \approx 1-(1-\frac{1}{2}x^2) = x^2/2$

hence $\frac{1-\cos x}{x^2}$ is bounded

∘ near ∞ : $\frac{1-\cos x}{x^2} < \frac{2}{x^2}$, $\int_1^\infty \frac{1}{x^2}\, dx = \int_1^\infty d\left(\frac{-1}{x}\right)$
$= -\frac{1}{x}\Big|_1^\infty < \infty$

in general $\int^\infty \frac{1}{x^p}\, dx$ convergent if $p>1$.

∘ Let I denote the integral on the left hand side (LHS)

$$I = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_\varepsilon^R \frac{1-\cos x}{x^2}\, dx$$

using $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, we have

$$I = \lim \int_\varepsilon^R \frac{2-(e^{ix}+e^{-ix})}{2x^2}\, dx$$

$$= \lim \int_\varepsilon^R \frac{1-e^{ix}}{2x^2}\, dx + \int_\varepsilon^R \frac{1-e^{-ix}}{2x^2}\, dx$$

let $u=-x$

$$= \lim \int_\varepsilon^R \frac{1-e^{ix}}{2x^2}\, dx + \int_{-R}^{-\varepsilon} \frac{1-e^{iu}}{2u^2}\, du$$

rename

rename u back to x

$$= \lim \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right)\left(\frac{1-e^{ix}}{2x^2}\right) dx$$

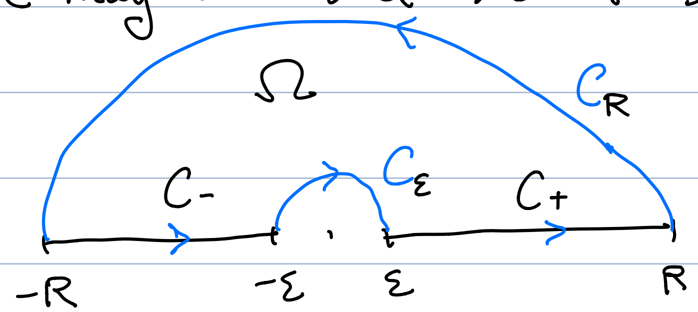
warning: $\int_{\varepsilon}^{1} \frac{1-e^{ix}}{x^2}\,dx \sim \int_{\varepsilon}^{1} \frac{1-(1-ix\cdots)}{x^2}\,dx$

$\sim \int_{\varepsilon}^{1} \frac{i}{x}\,dx \sim -i\ln\varepsilon$ is divergent as $\varepsilon \to 0$

However, the divergence cancels out with $\int_{-1}^{-\varepsilon} \frac{1-e^{ix}}{x}\,dx$.

Compare with the case of $\frac{1-\cos x}{x^2}$.

Now, we may add and subtract some auxillary arcs



$$f(z) = \frac{1-e^{iz}}{2z^2}$$

$$\left(\int_{C_-} + \int_{C_+} + \int_{C_\varepsilon} + \int_{C_R}\right) f(z)\,dz = 0$$

$\because$ $f(z)$ is holomorphic on the contour  and its interior

$\therefore$ by Cauchy theorem, the integral $= 0$.

$$\therefore \quad \left(\int_{C_-} + \int_{C_+}\right) f(z)\,dz = -\left(\int_{C_\varepsilon} + \int_{C_R}\right) f(z)\,dz.$$

We will show that $I_R = \int_{C_R} f(z)\,dz \sim O\left(\frac{1}{R}\right) \xrightarrow{R\to\infty} 0$

$$I_\varepsilon = \int_{C_\varepsilon} f(z)\,dz = -\frac{\pi}{2} + O(\varepsilon) \longrightarrow -\frac{\pi}{2} \quad \text{as } \varepsilon \to 0.$$

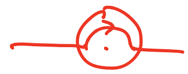
• $I_\varepsilon = \int_{C_\varepsilon} f(z)\,dz = \int_{C_\varepsilon} \frac{1 - \left(1 + iz + \frac{(iz)^2}{2} + z^2\psi(z)\right)}{2z^2}\,dz$ $\psi(z)\to 0$ as $z \to 0$

$e^{iz} = 1 + iz + \frac{(iz)^2}{2} + z^2\cdot\psi(z)$

$= \int_{C_\varepsilon} \frac{-i}{2z}\,dz + \underbrace{\int_{C_\varepsilon} \overbrace{\frac{1}{4}\left(1 - \psi(z)\right)}^{\text{bounded by } C}\,dz}_{O(\varepsilon)}$ length$(C_\varepsilon) = \pi\varepsilon$

∵ integrand is bounded. length $\sim \varepsilon$.

we parametrize C_ε by $z = \varepsilon\cdot e^{i\theta}$, θ from π to 0



$= \int_\pi^0 \frac{-i}{2\varepsilon\cdot e^{i\theta}}\,\varepsilon\cdot e^{i\theta}\cdot i\,d\theta + O(\varepsilon)$

$= -\int_0^\pi \frac{1}{2}\,d\theta + O(\varepsilon) = -\frac{\pi}{2} + O(\varepsilon).$

$\theta \in [0, \pi]$

• For z on C_R, $z = R\cdot e^{i\theta} = R\cos\theta + iR\sin\theta$.

$e^{iz} = e^{iR\cos\theta - R\sin\theta} = e^{-R\sin\theta}\cdot e^{iR\cos\theta}$

$|e^{iz}| = e^{-R\sin\theta} \le e^0 = 1$ ∵ $R\sin\theta \ge 0$ ("$\mathrm{Re}(z)$") for $\theta \in [0, \pi]$.

∴ $\left|\frac{1 - e^{iz}}{2z^2}\right| \le \frac{1 + |e^{iz}|}{2|z|^2} \le \frac{1}{R^2}$. $|a \pm b| \le |a| + |b|$

$\left|\int_\gamma f(z)\cdot dz\right| \le \int_\gamma |f(z)|\cdot|dz| \le \max|f|\cdot\int_\gamma |dz|$

$\left|\sum f(z)\cdot\Delta z\right| \le \sum |f(z)\cdot\Delta z| = \sum |f(z)|\cdot|\Delta z| \le \sum C\cdot|\Delta z| = C\cdot L(\gamma)$

∴ $\left|\int_{C_R} f(z)\cdot dz\right| \le \max_{z\in C_R} |f(z)|\cdot \text{length } C_R$

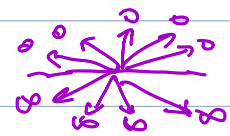
$\le \frac{1}{R^2}\cdot\pi R = O\left(\frac{1}{R}\right)$

∴ $I = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_{C_- + C_+} f(z)\,dz\right) = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(-\int_{C_\varepsilon + C_R} f(z)\,dz\right)$

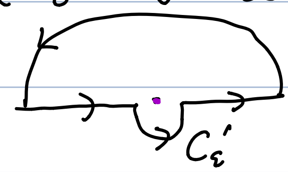
$= \frac{\pi}{2}$

Remark: • why we choose to "complete the contour" by adding the 2 arcs in that way? Any alternative approach?

• for C_R, we can only complete the contour by going in the upper half plane, to keep e^{iz} bounded

$\frac{1-e^{iz}}{2z^2}$  e^{iz}

- for C_ε, actually we can choose to add a small arc below $z=0$ (instead of above).

$\int$  $= 2\pi i \cdot \underbrace{\operatorname{Res}_{z=0} f(z)}_{=\text{coeff of } \frac{1}{z} \text{ term in Laurent exp.}} = 2\pi i \cdot \left(\frac{-i}{z}\right) = \pi$

$\int_{C_\varepsilon'} = \frac{\pi}{2} + O(\varepsilon)$

$\therefore \int_{C_- + C_+} = \int - \int = \pi - \frac{\pi}{2} = \frac{\pi}{2}$ same answer.

$e^{iz} = e^{i(x+iy)} = e^{ix-y} = e^{-y} \cdot e^{ix}$

2. Jordan Lemma (useful for Ex 2 in Homework).

Let $Q(z)$ be a complex valued function on the upper half plane. If $|Q(z)| \to 0$ uniformly as $|z| \to \infty$, for $0 \le \arg(z) \le \pi$, then

$$\lim_{R\to\infty} \int_{C_R} Q(z) \cdot e^{ipz}\, dz = 0$$

$\left| \int_{C_R} \frac{e^{iz}}{z} dz \right| \le \max_{z\in C_R} \left| \frac{e^{iz}}{z} \right| \cdot L(C_R) \le \frac{1}{R} \cdot R = 1$

for $p > 0$. Here C_R is the upper semi-circle with radius R, counter clockwise orientation. $z = Re^{i\theta}, 0 \le \theta \le \pi$.

$|Q(z)| \to 0$ uniformly in $0 \le \arg(z) \le \pi$ as $|z| \to \infty$ means, $\forall \varepsilon > 0$, $\exists R > 0$, s.t for any $z \in \mathbb{C}$ with $0 \le \arg z \le \pi$. and $|z| > R$. $|Q(z)| < \varepsilon$.

uniform means independent of $\arg(z)$.

$h_n(\theta) = |Q(n \cdot e^{i\theta})|$, $g_\theta(r) = |Q(r \cdot e^{i\theta})|$. $\theta \in [0, \pi]$. uniform w.r.t. θ

Pf: when $z \in C_R$, let $z = R \cdot e^{i\theta}$. then

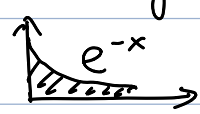
$$\left| \int_{C_R} Q(z) \cdot e^{ipz} dz \right| = \left| \int_0^\pi Q(Re^{i\theta}) \cdot e^{ipR(\cos\theta + i\sin\theta)} Re^{i\theta} i \, d\theta \right|$$

$$\le \int_0^\pi |Q(z)| \cdot \left| e^{ipR\cos\theta - pR\sin\theta} \right| \cdot R \cdot d\theta$$

$$\le \max_{z \in C_R} |Q(z)| \cdot R \cdot \underbrace{\int_0^\pi e^{-pR\sin\theta} d\theta}$$

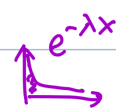
It is important to estimate the integral sharply.

$\boxed{\text{idea:}}$ $\int_0^\infty e^{-x} dx = 1$

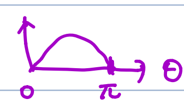


exponential decay.

$\lambda > 0$. $\int_0^\infty e^{-\lambda x} dx = \frac{1}{\lambda}$ integral receive contribution mainly for $x \lesssim \frac{1}{\lambda}$.



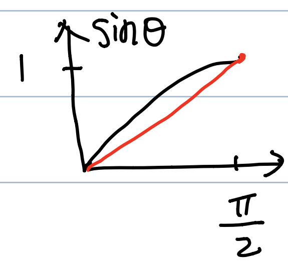
$\therefore$ near $\theta = 0$, $\sin\theta \approx \theta + O(\theta^3)$



near $\theta = \pi$, let $\theta = \pi - \varphi$, $\sin(\pi - \varphi) = \sin\varphi = \varphi + O(\varphi^3)$

$\therefore$ integral of $\int_0^\pi e^{-PR\sin\theta} d\theta \sim 2\int_0 e^{-PR\theta} d\theta \sim \frac{1}{PR}$

More rigorously; we have



$\sin\theta \ge \theta/(\pi/2)$ for $\theta \in [0, \frac{\pi}{2}]$.

$$\therefore \int_0^{\pi} e^{-PR\sin\theta}\, d\theta = 2\cdot \int_0^{\frac{\pi}{2}} e^{-PR\sin\theta}\, d\theta$$

$$\leq 2\cdot \int_0^{\frac{\pi}{2}} e^{-PR\cdot\frac{2}{\pi}\theta}\, d\theta \leq 2\cdot \int_0^{\infty} e^{-PR\cdot\frac{2}{\pi}\theta}\, d\theta$$

$$= 2\cdot \frac{\pi}{2PR} = O\left(\frac{1}{R}\right)$$

$$\therefore \left| \int_{C_R} Q(z)\, e^{iPz}\, dz \right| = o(1)\cdot R\cdot O\left(\frac{1}{R}\right) = o(1)$$ #

Little O, Big-O notation. Say $a(R)$, $b(R)$ are function of R. as $R\to\infty$

- We write "$a(R) = O(b(R))$", if $\exists R_0, C > 0$, such that for all $R > R_0$, $|a(R)| \leq C\cdot|b(R)|$.

$$\Longleftrightarrow \limsup_{R\to\infty} \frac{|a(R)|}{|b(R)|} < \infty$$

- we write "$a(R) = o(b(R))$", if $\forall \varepsilon > 0$. $\exists R_0 > 0$, such that $\forall R > R_0$. $|a(R)| < \varepsilon\cdot|b(R)|$.

i.e. $\limsup_{R\to\infty} \frac{|a(R)|}{|b(R)|} = 0$.

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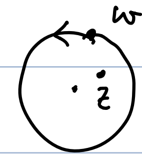
in hw #2

$$\int_{C_R} \frac{e^{iz}-1}{z}\, dz = \underbrace{\int_{C_R} \frac{e^{iz}}{z}\, dz}_{\to 0} + \underbrace{\int_{C_R} \frac{-1}{z}\, dz}_{\to \neq 0.}$$

Thm :
$$f(z) = \oint_{|w|=1} \frac{f(w)}{w-z} \frac{dw}{2\pi i}$$



$f(z)$ is hol'c on $\overline{\mathbb{D}}$

$z \in \mathbb{D}$

Cor :
$$f^{(n)}(z) = \oint_{|w|=1} \left(\frac{\partial}{\partial z}\right)^n \left(\frac{f(w)}{w-z}\right) \frac{dw}{2\pi i}$$
$$= \oint_{|w|=1} (n!) \cdot \frac{f(w)}{(w-z)^{n+1}} \frac{dw}{2\pi i}$$

Pf : We prove by induction. Say for $n-1$, it works.

Then
$$\frac{\partial}{\partial z}\left(f^{(n-1)}(z)\right) = \lim_{h\to 0} \frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h}$$

$$f^{(n-1)}(z+h) - f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \oint \frac{f(w)}{(w-(z+h))^n} - \frac{f(w)}{(w-z)^n} \cdot dw.$$

(smart)

- $$A^k - B^k = (A-B)\underbrace{(A^{k-1} + A^{k-2}\cdot B + \cdots + B^{k-1})}_{k \text{ terms.}}$$

if $A = \frac{1}{w-(z+h)}$, $B = \frac{1}{w-z}$. $k=n$

then
$$A - B = \frac{h}{(w-(z+h))(w-z)} , \quad A^{k-1-i} B^i \xrightarrow{h\to 0} \left(\frac{1}{w-z}\right)^{k-1}.$$

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{(n-1)!}{2\pi i} \oint . f(w) \cdot \frac{1}{h} \cdot \frac{h}{(w-(z+h))(w-z)} \cdot \overbrace{(\cdots\cdots)}^{n\text{-terms.}} dw$$

as $h \to 0$.

$$\xrightarrow[\text{uniformly converge.}]{\because \text{ integrand}} \frac{(n-1)!}{2\pi i} \cdot \oint f(w) \frac{1}{(w-z)^2} \cdot \frac{n}{(w-z)^{n-1}} \cdot dw.$$

$$= \frac{n!}{2\pi i} \cdot \oint f(w) \frac{1}{(w-z)^{n+1}} \cdot dw.$$

(brutal force approach). $\frac{1}{w-(z+h)} = \frac{1}{w-z-h} = \frac{1}{w-z}\frac{1}{1-\frac{h}{w-z}}$

$$= \frac{1}{w-z}\left(1+\frac{h}{w-z}+O(h^2)\right).$$

$$\left(\frac{1}{w-(z+h)}\right)^n = \frac{1}{(w-z)^n}\left(1+\frac{nh}{w-z}+O(h^2)\right).$$