HW 6: Meromorphic Functions

1. Essential singularities.

Consider the function $f(z) = e^{z^3}$.

1. For $\theta \in [0, 2\pi)$, let $z = re^{i\theta}$. As $r \to \infty$, for which values of θ does $|f(z)|$ goes to ∞, goes to zero, or oscillate? Does your answer change if we change $f(z)$ to $P(z)e^{z^3}$ for a polynomial $P(z)$?
2. Is $z = \infty$ is an essential singularity of $f(z)$? Prove your answer.

2. Rational Function

Let $f(z) = \frac{(z-1)(z-3)}{(z-2)}$, and we view f as a function on the extended complex plane $\widehat{\mathbb{C}}$.

1. How many zeros and poles in $\widehat{\mathbb{C}}$ are there for f?
2. Write down the Taylor or Laurent expansion of f near $z = 1$ and $z = \infty$ (or $w = 0$, where $w = 1/z$). You only need to write down the first two terms.

3. Winding number

Recall that the winding number of a curve γ around a point z_0 (which does not lie on the curve) is defined as

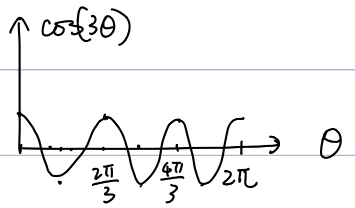
$$n(\gamma, z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0} dz.$$

Find the winding number of the following γ around $z_0 = 0$.

1. Let $f(z) = z^3 + z/2$. Let γ be the image of the unit circle under $f(z)$.
2. Let γ be image of the unit circle (counter-clockwise oriented) under the map $f(z) = (z - 1/2)^2/(z + 1/2)^2$.

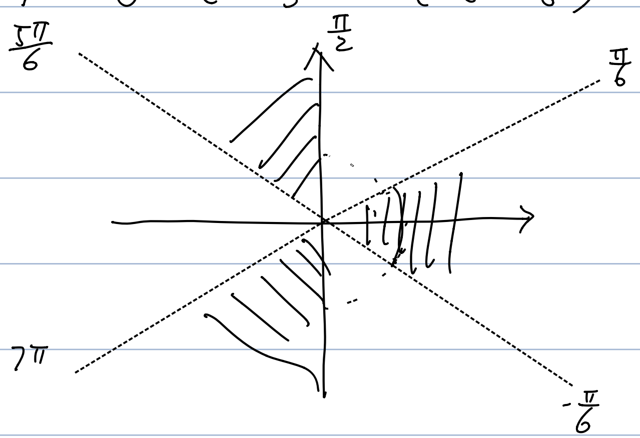
1. $|e^{z^3}| = e^{Re(z^3)}$, let $z = e^{3\rho + 3i\theta}$, then $z^3 = e^{3\rho + 3i\theta}$.

$Re(e^{3\rho+3i\theta}) = e^{3\rho} \cdot \cos(3\theta)$. $\therefore |e^{z^3}| = e^{e^{3\rho}\cdot\cos(3\theta)}$



so, for $3\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) + 2\pi n$, $\cos(3\theta) > 0$

$\iff \theta \in \frac{2\pi}{3}n + (-\frac{\pi}{6}, \frac{\pi}{6})$, $\cos\theta\,(3\theta) > 0$.



one divide 2π into 6 even portions, the shaded region has $\cos(3\theta) > 0$ and the unshaded region has $\cos(3\theta) < 0$.

$|f(z)|$ grows as $|z| \to \infty$ with $\cos(3\theta) > 0$,

$|f(z)|$ decays as $|z| \to \infty$ with $\cos(3\theta) < 0$.

$|f(z)|$ oscillates ———— if $\cos(3\theta)=0$.

The asymptotic behavior will not change if we consider $P(z)e^{z^3}$ since there is an $R>0$, and $C>0$, s.t. for all $|z|>R$,

$\frac{1}{C}\cdot|z|^n \leq |P(z)| \leq C\cdot|z|^n$, where $n=\deg(P)$.

For any fixed θ,

- if $\cos 3\theta>0$

$$\lim_{\substack{z=e^{\rho+i\theta}\\ \rho\to\infty}}|P(z)\cdot e^{z^3}| \geqslant \lim_{\rho\to\infty}\frac{1}{C}(e^{\rho})^n\cdot e^{e^{3\rho}\cos 3\theta} = \frac{1}{C}\lim_{\rho\to\infty} e^{e^{3\rho}\cos 3\theta+3\rho}$$
$$=\infty$$

- if $\cos 3\theta<0$

$$\lim_{\rho\to\infty}|P(z)\cdot e^{z^3}| \leqslant \lim_{\rho\to\infty} C\cdot e^{\rho n}\cdot e^{e^{3\rho}\cdot\cos 3\theta} = C\cdot\lim_{\rho\to\infty} e^{e^{3\rho}\cos 3\theta+n\cdot\rho} = 0$$

as $\lim_{\rho\to\infty}(e^{3\rho}\cos 3\theta+n\cdot\rho)=-\infty$ since $\cos(3\theta)<0$.

- if $\cos 3\theta=0$, then $|e^{z^3}|=1$, and $\lim_{|z|\to\infty}|e^{z^3}\cdot P(z)|=\lim_{|z|\to\infty}|P(z)|=\infty$

So the grow or decay behavior is roughly unchanged, except in the interface where oscillation occurs.

(2) $z=\infty$, (or as $w=\frac{1}{z}$, $w=0$) is an essential singularity, since as $|z|\to\infty$, $|f(z)|$ is neither bounded or simply go to ∞, hence $z=\infty$ is neither a removable singularity nor a pole, hence $z=\infty$ is an essential singularity.

2. $f(z)=\frac{(z-1)(z-3)}{(z-2)}$. On $\hat{\mathbb{C}}$, $f(z)$ has

zeros : $z=1, \quad z=3$

poles : $z=2, \quad z=\infty.$

each of them has order 1.

- Near $z=1$, we have Taylor expansion.

let $u=z-1$, then

$$f(1+u) = \frac{(1+u-1)(1+u-3)}{(1+u-2)} = u\cdot\frac{(u-2)}{(u-1)}$$

$$= u\cdot(2-u)(1-u)^{-1}$$

$$= u\cdot(2-u)(1+u+u^2+\cdots)$$

$$= u\left((2+2u+2u^2\cdots) - u(1+u+u^2\cdots)\right)$$

$$= u(2+u+\cdots)$$

$$= 2u+u^2+\cdots.$$

- Near $z=\infty$, use $w=1/z$, and expand around $w=0$.

$$\tilde{f}(w) = f\left(\frac{1}{w}\right) = \frac{(1/w-1)(\frac{1}{w}-3)}{(\frac{1}{w}-2)} = \frac{(1-w)(1-3w)}{(1-2w)w}$$

$$= \frac{1}{w}\left((1-w)(1-3w)(1+2w+\cdots)\right)$$

$$= \frac{1}{w}(1-2w+\cdots)$$

$$= \frac{1}{w}-2+\cdots$$

#3: Find the winding number around 0 for the following curve:

(1) $\gamma = f(C)$, C = unit circle, $f(z) = z^3+\frac{1}{2}z$

(2) γ = - — — — -, $f(z) = \dfrac{(z-\frac{1}{2})^2}{(z+\frac{1}{2})^2}.$

Sol'n: the winding number is

(1) $$n(\gamma, 0) = \frac{1}{2\pi i} \int_{w \in f(C)} \frac{1}{w-0}\, dw = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz$$

$$= (\text{number of zeros of } f(z) \text{ inside } C) - (\#\text{ of poles of } f \text{ inside } C)$$

$$= 3 - 0 = 3.$$

since the roots are $z = 0$, $z = \pm\sqrt{\frac{1}{2}}$.

(2) $$n(\gamma, 0) = (\#\text{ zero} - \#\text{ of pole}) \text{ of } f \text{ in } C$$

$$= 2 - 2 = 0.$$

4. Meromorphic Function on a disk

In this problem, we are going to construct meromorphic function on the unit disk with infinitely many poles. In class, I give the example of $f(z) = 1/\sin(1/(z-1))$. In this problem, we are going to do another construction: show that

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n}{(z - \frac{n-1}{n})}$$

is a meromorphic function in $\mathbb{D}$ with poles $1 - 1/n$, if $\sum_n |a_n| < \infty$.

5. The "Failure" of Cauchy integral formula

Recall that if f is a holomorphic function on the unit circle and its interior, we can reconstruct the value of $f(z)$ for $|z| < 1$ from the boundary value at $\{|z| = 1\}$

$$f(z) = \frac{1}{2\pi i} \int_{|w|=1} \frac{f(w)}{w - z} dw.$$

One mischievous student in complex analysis wants throw a crank in the machine, and ask “what if we use an arbitrary complex valued function on the unit circle as input for the Cauchy integral formula? ”. Your job is to investigate what 'disaster' will follow if he does so. Here is the mission description:

- Let $g(w)$ be a smooth **complex valued** function, only defined on the unit circle $|w| = 1$.
- Define a function $f(z)$ using Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{|w|=1} \frac{g(w)}{w - z} dw.$$

Question:

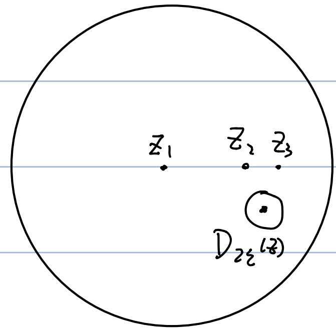
1. Show that $f(z)$ is a holomorphic function for $\{|z| < 1\}$ and $\{|z| > 1\}$. Show that $f(z) \to 0$ as $z \to \infty$.
2. Let C be the unit circle, for any $z_0 \in C$, does

$$\lim_{z \to z_0} f(z) = g(z_0)?$$

3. If you only know $f(z)$ inside unit disk $\{|z| < 1\}$, can you reverse-engineer to get the function $g(z)$ on the unit circle?

#4: For any point $z \in \mathbb{D}$ away from the poles, we have a small disk $D_\varepsilon(z)$ that is free from poles. $\{z_1, z_2, \cdots\}$

$z_n = \frac{n-1}{n}$.



poles:

$z_1 = 0$

$z_2 = \frac{1}{2}$

$z_3 = \frac{2}{3}$

$\vdots$

Then for any $z' \in \overline{D_\varepsilon(z)}$, such that

$|z' - z_n| > \varepsilon$.

$A = \sum_n |a_n|$

Then $|f(z')| \leq \sum_{n=1}^{\infty} \frac{|a_n|}{|z'-z_n|} \leq \frac{1}{\varepsilon} \sum_{n=1}^{\infty} |a_n| \lesssim \frac{A}{\varepsilon}$.,

And $|f'(z')| \leq \sum_{n=1}^{\infty} \cdot \frac{|a_n|}{|z'-z_n|^2} \leq \frac{1}{\varepsilon^2} \cdot A < \infty$

Hence f is holomorphic in a neighborhood $D_\varepsilon(z)$ of z,

for any $z \in \mathbb{D} \setminus \{z_1, z_2, \cdots, z_n\}$.

#5: (1) Let $F(\theta, z) = \frac{g(e^{i\theta})}{e^{i\theta} - z} \frac{e^{i\theta} \cdot i}{2\pi i}$ be a function

on $[0, 2\pi] \times \mathbb{D}$, then by Thm 5.2 in Ch 2. we know

$$f(z)\ \frac{1}{2\pi i} \int \frac{g(e^{i\theta})}{e^{i\theta} - z}\, de^{i\theta} = \frac{1}{2\pi i} \int_{\theta=0}^{2\pi} \frac{g(e^{i\theta})}{e^{i\theta} - z} \cdot e^{i\theta} \cdot i \, d\theta$$

$$= \int_0^{2\pi} F(\theta, z)\, d\theta.$$

is a holomophic function on $\mathbb{D}$,

Similarly, view $F(\theta, z)$ as a function on

$[0, 2\pi] \times (\mathbb{C} \setminus \overline{\mathbb{D}})$ is hol'c in z, continuous in

both variables., hence $f(z)$ is also hol'c on $\{|z| > 1\}$.

Let $C_1 = \max_{|z|=1} |g(z)|$, then for any $|z| > 1$, we have

$$|f(z)| \leq \left(\sup_{|w|=1} \frac{|g(w)|}{|w-z|} \right) \cdot \frac{\text{Length } C}{2\pi} \leq \frac{C_1}{|z|-1} \to 0 \quad \text{as } |z| \to \infty.$$

(2) and (3). The story goes as follows:

We first consider $g(z) = z^n$ for $z \in C$ (unit circle), $n \in \mathbb{Z}$.

then for

$$n \geq 0, \quad f(z) = \begin{cases} z^n & |z| \leq 1 \\ 0 & |z| > 1 \end{cases}$$

$$n < 0 \quad f(z) = \begin{cases} 0 & |z| \leq 0 \\ -z^n & |z| > 1 \end{cases}$$

In general, for $z_0 \in \mathbb{C}$, we have the difference of the two boundary values from inside and outside being $g(z)$:

$$\lim_{r \to 1^-} f(r \cdot z_0) - \lim_{r \to 1^+} f(r \cdot z_0) = g(z_0)$$

This is clear if $g(z)$ is a finite linear combination of z^n, $g(z) = a_1 \cdot z^{n_1} + a_2 \cdot z^{n_2} + \cdots + a_N \cdot z^{n_N}$.
This is also true more generally, for any $g(z)$ smooth function on C, or even just continuous function.

So the answer for both (2) and (3) is <u>No</u>.