1. If $f: \mathbb{D} \to \mathbb{D}$ is hol'c, and $f(0) = 0$, show that $|f(z)| < |z|$ for all $|z| < 1$.

Pf: Let $g(z) = \begin{cases} f(z)/z & z \neq 0,\ z \in \mathbb{D} \\ f'(0) & z = 0. \end{cases}$

then $g(z)$ is a hol'c function in $\mathbb{D} \setminus \{0\}$ and continuous at $z = 0$. By removable singularity thm, $g(z)$ is hol'c in $\mathbb{D}$.

For any $0 < r < 1$, we let $M(r) = \sup_{|z| < r} |g(z)|$. By Stein corollary 4.6,

$$M(r) = \sup_{|z| = r} |g(z)| = \sup_{|z| = r} \frac{|f(z)|}{|z|} = \frac{1}{r} \sup_{|z| = r} |f(z)|$$

$$\leq \frac{1}{r}.$$

And, if $r_1 \leq r_2$, $M(r_1) = \sup_{|z| < r_1} |g(z)| \leq \sup_{|z| < r_2} |g(z)| = M(r_2)$.

Hence, $\forall\, r < 1$, $M(r) \leq M(r') \leq \frac{1}{r'}$ for all $r' > r$.

Take limit $r' \to 1$, we get $M(r) \leq 1$ for all $r < 1$.

Hence

$$\sup_{|z| < 1} |g(z)| = \sup_{r < 1} M(r) \leq 1.$$

Thus, $\forall\, |z| < 1$, $|f(z)| \leq |z|$.

2. How many solution does $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $|z| < 1$?

Ans: 3. By Rouché thm, let $f(z) = 6z^3$,

$g(z) = z^7 - 2z^5 - z + 1$, then on $|z|=1$, we have

$|f(z)| = 6$, $|g(z)| \le |z^7| + |2z^5| + |z| + 1 = 5$.

hence $|f(z)| > |g(z)|$ on $|z|=1$. Thus

$f(z)+g(z)$ has the same number of roots as $f(z)$

in D, namely 3.

3. How many roots does $z^4 - 6z + 3 = 0$ have

on the annulus $1 < |z| < 2$?

Ans: we will count roots inside $|z|<2$ and $|z|<1$.

For $|z|<2$, we let $f(z) = z^4$, $g(z) = -6z+3$. Then

on $|z|=2$, $|f(z)| = 16$, $|g(z)| \le |-6z| + 3 = 15$. , hence.

$f(z)+g(z)$ has the same number of roots as $f(z)$

inside $|z|=2$, namely 4.

For $|z|<1$, let $f(z) = -6z$, $g(z) = z^4+3$, then

on $|z|$, $|f(z)| = 6$, $|g(z)| \le 4$. we have

number of roots of $f+g$ equal to that of f inside

$|z|=1$, namely 1.

Hence, # of roots of $z^4 - 6z + 1$ has $4-1=3$ roots

in the annulus $1 < |z| < 2$.

#4: Are the following open maps?

1) $f: \mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z}$.

yes, since complex conjugation is a linear isomorphism hence is open.

2) $f: \mathbb{C} \to \mathbb{R}$, $f(z) = |z|^2$.

Not open. for any open ball $B_r(0)$,

$f(B_r(0)) = [0, r^2)$, which is not open.

3) $f: \mathbb{C} \to \mathbb{R}$. $f(x+iy) = x \cdot y$.

Yes. $f(z) = \mathrm{Im}(z^2/2)$, hence f is a composition of a holomorphic map $\mathbb{C} \to \mathbb{C}$, with a projection $\mathbb{C} \to \mathbb{R}$, both are open map, hence f is open

4) $f: \mathbb{C} \to \mathbb{R}$ $f(z) = \mathrm{Re}(z^3 + 2z)$.

open. same reason as 3).

5. Let $\Omega = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

1) let $\gamma_1 : [0,1] \to \mathbb{C}^*$ be the unit circle

$t \mapsto e^{2\pi i \cdot t}$

and let $\gamma_0 : [0,1] \to \mathbb{C}^*$ be constant path

$\gamma_0(t) = 1$.

Show that γ_1 is not homotopic to γ_0 in Ω.

2) Show that, for any $\gamma : [0,1] \to \mathbb{C}^*$, s.t. $\int_\gamma \frac{1}{z}\, dz = 0$, $\gamma(0) = \gamma(1) = 1$ γ is homotopic to the constant path.

Pf: (1) Consider $\int \frac{1}{z}\, dz$ along γ_0 and γ_1, we have

$$\int_{\gamma_1} \frac{1}{z}\, dz = 2\pi i, \qquad \int_{\gamma_0} \frac{1}{z}\, dz = 0.$$

Hence they are not homotopic.

2) We construct a continuous function $\tilde{\gamma} : [0,1] \to \mathbb{C}$, s.t.

$$\begin{cases} \tilde{\gamma}(0) = 0, \\ e^{\tilde{\gamma}(t)} = \gamma(t). \end{cases}$$

Indeed, this can be achieved, by letting

$\tilde{\gamma}(t) = \rho(t) + i\,\theta(t)$. $\rho(t), \theta(t) \in \mathbb{R}$.

and $\rho(t) = \log |\gamma(t)|$, $\theta(t) \equiv \arg(\gamma(t)) \mod 2\pi$.

and $\theta(t)$ is continuous in t.

Since $$\frac{\theta(1) - \theta(0)}{2\pi} = n(\gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z}\, dz = 0.$$

we have $\theta(1) = \theta(0) = 0$ Also $\rho(1) = \rho(0) = 0$ by construction.

Hence, $\tilde{\gamma} : [0,1] \to \mathbb{C}$ is a loop with endpoints $\tilde{\gamma}(0) = \tilde{\gamma}(1) = 0$.

We construct a homotopy from $\tilde{\gamma}$ to the constant map to 0 by linear interpolation:

$$\tilde{\gamma}_s(t) = (1-s)\,\tilde{\gamma}(t).$$

Then, we define

$$\gamma_s(t) = e^{\tilde{\gamma}_s(t)} = e^{(1-s)\tilde{\gamma}(t)}$$

This is a homotopy from γ to constant map to 1. $\square$.