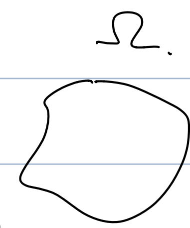
2.5.2 : Uniform Convergence of Holomorphic functions preserve holomorphicity, and derivatives also converges locally uniformly.

2.5.3 : Weighted sum / integral of holomorphic function is holomorphic.

2.5.4 : Reflection principle : (if $f(z)$ is real when $z \in \mathbb{R}$ ····)

2.5.5 : Runge Approximation Theorem (approximations to approximations ····)

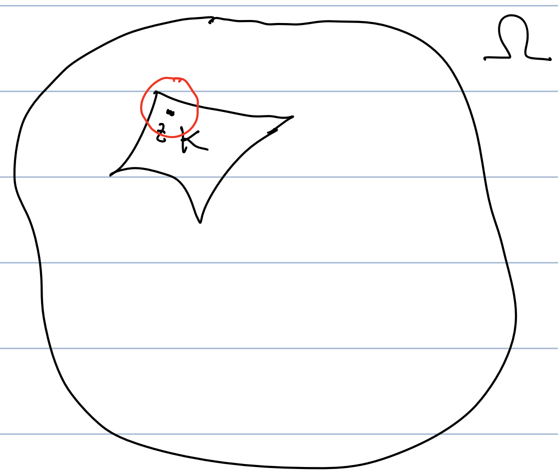
Let $\{f_n\}$ be a sequence of hol'c functions on Ω.
For any compact subset of Ω, f_n converges uniformly to f. Then $\{f_n'\}$ also converges "locally" uniformly on Ω.
("for any compact subset").



Pf :

• Ω open.

• $K \subset \Omega$ compact



• Let $\delta = \min\limits_{z \in K} \operatorname{dist}(z, \partial\Omega)$. Let $r = \frac{1}{2}\delta$, then.

for any $z \in K$, $\overline{D_r(z)} \subset \Omega$

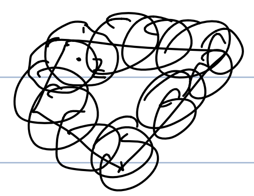
- $f'(z) = \oint_{\partial D_r(z)} \frac{f(w)}{(w-z)^2} \frac{dw}{2\pi i}$, $\quad f_n'(z) = \oint_{\partial D_r(z)} \frac{f_n(w)}{(w-z)^2} \frac{dw}{2\pi i}$

- <u>Goal</u>: show $f_n' \to f'$ uniformly on K.

$$\left| f_n'(z) - f'(z) \right| = \left| \oint_{\partial D_r(z)} \frac{f_n(w) - f(w)}{(w-z)^2} \frac{dw}{2\pi i} \right|.$$

$K + \bar{D}_r \subset \Omega$, compact.

$$\leq \oint_{\partial D_r(z)} \frac{|f_n(w) - f(w)|}{|w-z|^2} \cdot \frac{|dw|}{2\pi}.$$



$$\leq \underbrace{\left(\sup_{w \in \underline{K + \bar{D}_r}} |f_n(w) - f(w)| \right)}_{\downarrow \text{ as } n\to\infty \\ 0} \cdot \underbrace{\frac{1}{r^2} \cdot \frac{2\pi r}{2\pi}}_{= \frac{1}{r}}$$

Minkowski sum. "+"

$A, B \subset \mathbb{C}$. $A+B = \{ a+b \mid a \in A, b \in B \}$.

$\therefore \quad |f_n'(z) - f'(z)| \to 0$ uniformly on K. #

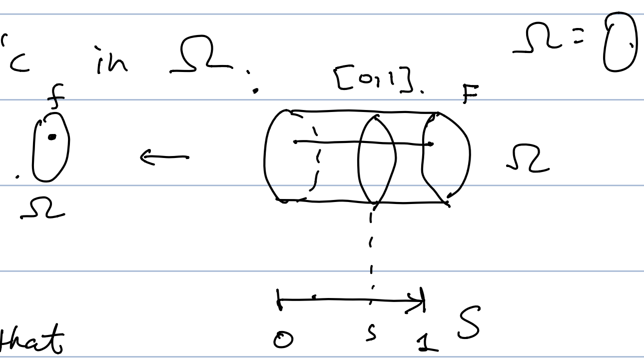
§5.3. Holomorphic function defined ~~into~~ in terms of integrals

<u>Thm</u> (5.4) Let $F(z,s)$ be defined for $\Omega \times [0,1]$. $\Omega \subset \mathbb{C}$ open. Suppose that

- $F(z,s)$ is hol'c in z for each fixed s.
- $F(z,s)$ is continous on $\Omega \times [0,1]$.

then

$$f(z) = \int_0^1 F(z,s)\, ds \quad \text{is hol'c in } \Omega.$$



<u>Pf</u>: Use Morera theorem, suffice to test on any triangle T in Ω, that

$$\int_T f(z) \cdot dz = 0.$$

$$\Leftrightarrow \quad \int_T \int_0^1 F(z,s)\, ds\, dz = 0.$$

$\because$ $\displaystyle\int_T \int_0^1 |F(z,s)|\, ds \cdot |dz|$

$\displaystyle\leq \sup_{\substack{z\in T\\ s\in[0,1]}} |F(z,s)| \cdot \underbrace{\iint_T 1 \cdot ds\, |dz|}_{= 1 \times \text{length}(T).}$

$< \infty$

- From real analysis / measure theory

$$\iint g(x,y)\, dx\, dy = \iint g(x,y)\, dy\, dx.$$

if $\displaystyle\iint |g(x,y)|\, dx\, dy < \infty.$

- compare

$$\sum_n \sum_m a_{n,m} = \sum_m \sum_n a_{n,m}$$

if $\displaystyle\sum_{n,m} |a_{n,m}| < \infty.$

$\because$ $F(z,s)$ is continuous on $\Omega \times [0,1]$

$T \times [0,1] \subset \Omega \times [0,1]$ is a closed & bounded in $\mathbb{C}\times[0,1]$.

$\Rightarrow$ $T \times [0,1]$ is compact.

$\therefore$ $|F(z,s)|$ on a compact set has a maximum.

$$\therefore \int_T \int_0^1 F(z,s)\, ds\, dz = \int_0^1 \int_T F(z,s)\, dz \cdot ds = \int_0^1 0 \cdot ds = 0.$$

#.

Linear combination : if V is a vector space,

- $v_1, v_2 \in V$, $av_1 + bv_2$ is a linear combination
- v_s for $s \in [0,1]$, changing continuous w.r.t. s.

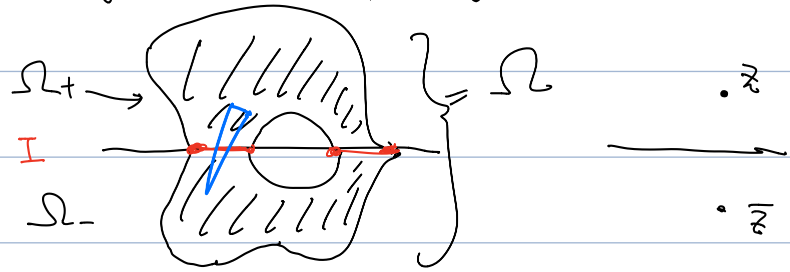
$\displaystyle\int_0^1 v_s\, ds.$ is a "linear combination" of vectors

- In this theorem : V = function on Ω.

5.4 Schwarz Reflection Principle.

Goal : to extend the domain of a holomorphic function.

$$\begin{cases} \Omega = \Omega_+ \sqcup \Omega_- \sqcup I \\ \partial\Omega_+ \cap \partial\Omega_- = I. \\ \overline{\Omega_+} = \Omega_- \end{cases}$$



Thm: if f is a holomorphic function on Ω_+, and f can be extended to a continuous function on $\overline{\Omega_+}$, such that $f(x) \in \mathbb{R}$ for $x \in I$.

· then. the following function is holomorphic

$$F(z) = \begin{cases} f(z) & z \in \Omega_+ \\ f(z) = \overline{f(\bar{z})} & z \in I \\ \overline{f(\bar{z})} & z \in \Omega_- \end{cases}$$

Pf: ① $f_-(z) = \overline{f(\bar{z})}$ for $z \in \Omega_-$ is holomorphic.

· known: $f(x+iy) = u + iv$. $\begin{cases} \partial_x u = \partial_y v \\ \partial_x v = -\partial_y u \end{cases}$

· $f_-(\alpha + i\beta) = \tilde{u}(\alpha, \beta) + i\tilde{v}(\alpha, \beta)$.

$\alpha + i\beta = \overline{x+iy} \Rightarrow \alpha = x, \ \beta = -y$.

$f_- = \bar{f}$ $\quad \tilde{u} + i\tilde{v}\Big|_{(\alpha,\beta)} = \overline{u+iv}\Big|_{(x,y)}. \Rightarrow \tilde{u} = u, \ \tilde{v} = -v$

· $\tilde{u}(\alpha, \beta) = u(x,y) = u(\alpha, -\beta)$

· $\tilde{v}(\alpha, \beta) = -v(x,y) = -v(\alpha, -\beta)$

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial \alpha} = \dfrac{\partial \tilde{v}}{\partial \beta} \\ \dfrac{\partial \tilde{u}}{\partial \beta} = -\dfrac{\partial \tilde{v}}{\partial \alpha} \end{cases}$$

CR condition for $\tilde{u}, \tilde{v}$: follows from CR for u, v.

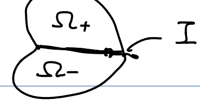
② $F(z)$ is a continuous function on Ω.

$\because$ if $z = x \in I$. $\quad f(\bar{z}) = f(x)$.

$\overline{f(\bar{z})} = \overline{f(x)} = f(x) = f(z)$.

$\because f(x)$ is real.

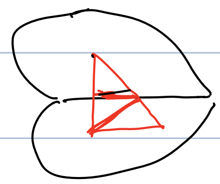
$\therefore$ on I, $F(z)$ is continuous.



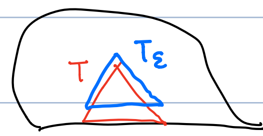
③. To chek F is hol'c, just need check for all triangles T in Ω,

$$\int_T F\,dz = 0.$$

- if T is in $\Omega_+ \sqcup I$ or $\Omega_- \sqcup I$, then ✓.
- if T intersects both Ω_+ and Ω_-, we can triangulate T into smaller pieces so that each piece is contained in $\Omega_+ \sqcup I$ or $\Omega_- \sqcup I$, then ✓



- if $T \subset \Omega_+ \sqcup I$, $T \cap I \neq \phi$.



then consider a slightly shifted T.

$T_\varepsilon = T + i\varepsilon$, then $T_\varepsilon \subset \Omega_+$

and $\int_{T_\varepsilon} F(z)\,dz = 0$. Then let $\varepsilon \to 0$, since F is a continuous function, $\int_T F(z)\,dz = \lim_{\varepsilon\to 0} \int_{T_\varepsilon} F(z)\,dz = 0$.

$$\int_{\gamma_\varepsilon} f(z)\cdot dz = \int_a^b \underbrace{f(\gamma_\varepsilon(t))\cdot\gamma_\varepsilon'(t)}_{\text{dep on } \varepsilon \text{ uniformly continuous.}}\,dt$$

$\gamma_\varepsilon : [a,b] \to \mathbb{C}$, changes with ε continuously.

$$\int_T F(z)\,dz - \int_{T_\varepsilon} F(z)\,dz = \int_T \left(F(z) - F(z+i\varepsilon)\right) dz.$$

$$\int_T |F(z) - F(z+i\varepsilon)|\,|dz| \leq \text{length}(T)\cdot \sup_{z\in T} |F(z) - F(z+i\varepsilon)|$$

$$\to 0 \quad \text{as } \varepsilon \to 0.$$

why we can "take limit in contours".