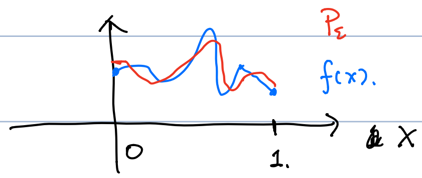
Runge Approximation Thm : approximate any holomorphic function on a compact set by a global rational function (or <u>sometimes</u> a polynomial).

· <u>Weierstrass Approximation Thm :</u>

Any continuous function f on closed interval $[0,1]$. can be <u>approximated uniformly</u> by polynomials.

i.e. $\forall\ \varepsilon > 0$, there is a $P_\varepsilon(x)$ polynomial, such that

$$\|f - P_\varepsilon\|_{[0,1]} := \sup_{x\in[0,1]} |f(x) - P_\varepsilon(x)| < \varepsilon.$$



vertical distance over any $x \in [0,1]$ is at most ε.

for fun:
(Look up a proof using Bernstein polynomial)

· <u>Motivation</u> : ① if f is a hol'c function on $\mathbb{D}$, then.
unit open disk.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for } z \in \mathbb{D}.$$

· at pointwise, for each fixed $z \in \mathbb{D}$, $\sum_{n=0}^{\infty} a_n \cdot z^n$ is a series of number., $P_N(z) = \sum_{n=0}^{N} a_n \cdot z^n$. $\lim_{N\to\infty} P_N(z) = f(z)$

∵ This is also true uniformly over any compact subset $K \subset \mathbb{D}$.

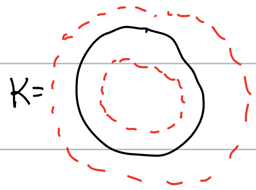
i.e. $\lim_{N\to\infty} \|f - P_N\|_K = 0$.

$P_N(z)$ is a sequence of <u>polynomials</u> that approximate f (uniformly over any compact subset in $\mathbb{D}$).

② Another example. Let $K = \{|z|=1\}$, unit circle. compact.

$f(z) = \frac{1}{z}$, for $z \in K$.

<u>Q</u>: Can one find a sequence of polynomials P_N, such that $\|P_N - f\|_K \to 0$ as $N \to \infty$?



f is given on an open nbhd Ω of K.

and f is hol'c on Ω.

$\Omega = \{ \frac{1}{2} < |z| < 2 \}$.

idea: • cut the circle and straighten it, to a straight interval, say, $[0, 2\pi] \in \mathbb{R}$. ....

Assume otherwise, $P_N \Rightarrow f$ uniformly on K,, then $C = \{|z|=1\}$.

<u>Ans</u>: <u>impossible.</u>. $\because \int_C f \cdot dz = \int_{|z|=1} \frac{1}{z}\, dz = 2\pi i$.

but $\int_C P_N(z)\, dz = 0$.

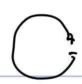
↑ polynomial.

$\lim_{N\to\infty} \int_C P_N\, dz \neq \int_C f\, dz$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$

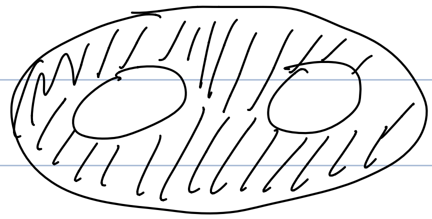
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \int_C \lim P_N\, dz$

contradiction.

<u>Rmk</u>: if K = "  " circle with an openning, then one can use <u>polynomial</u> to approximate f on K.

<u>Runge Thm</u>: Let $K \subset \mathbb{C}$ be a compact subsets, Let f be a hol'c function on an open neighborhood Ω of K. Then, f can be approximated by <u>rational function</u> ~~on~~ uniformly on K. Moreover, if K^c doesn't have any bounded component, then f can be approximated by a polynomial.

Ex:

$K =$ 

$K^c =$ 

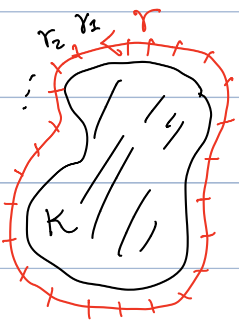
in this K^c has 3 connected components.

and 2 of them are bounded.

K^c has no bounded component $\Leftrightarrow$ K does not enclosed any hole in it

Sketch of idea: ① find a contour $\gamma \subset \Omega$ that enclose K

② $\forall z \in K$, $f(z) = \int_\gamma \frac{f(w)}{w-z} \frac{dw}{2\pi i}$ $\quad \delta_i = \int_{\gamma_i} dw$



$$= \sum_{i=1}^{N} \int_{\gamma_i} \frac{f(w)}{w-z} \frac{dw}{2\pi i} \approx \sum_{i=1}^{N} \frac{f(w_i)}{w_i - z} \cdot \frac{\delta_i}{2\pi i}$$

③ if K^c has no bounded component, then one can approximate $\frac{1}{z-z_0}$ $(z_0 \in K^c)$ by polynomials.

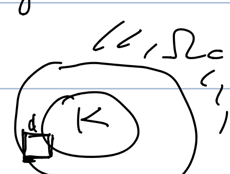
- a compact subset of $\mathbb{C}$ can be "ugly", say, Cantor set $\subset \mathbb{R} \subset \mathbb{C}$.

  ($\overline{0 \quad 1}$, remove the middle $1/3$ of each segment, repeat).

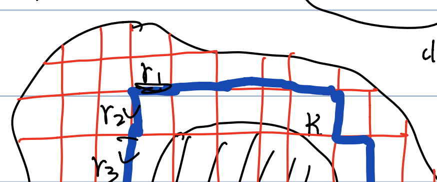
- consider $\{\frac{1}{n}, n = 1, 2, \cdots\} \cup \{0\} \subset \mathbb{C}$ $\quad 0 \cdots \frac{1}{3} \frac{1}{2}$

Step ①: Lemma: there exists a finite collection of segments. $\gamma_1, \cdots, \gamma_N$, in $\Omega \setminus K$, such that $\forall z \in K$,

$$f(z) = \frac{1}{2\pi i} \sum_{i=1}^{N} \int_{\gamma_i} f(w) \cdot dw.$$



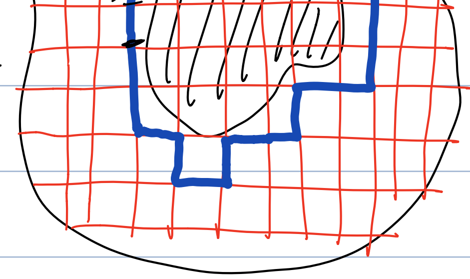
Pf: • $d = c \cdot \text{dist}(K, \Omega^c)$ (K compact, Ω^c closed) $\quad \cdot K \cap \Omega^c = \phi$



diameter(box) $\le \sqrt{2} d$ $<$ dist(K, Ω^c)

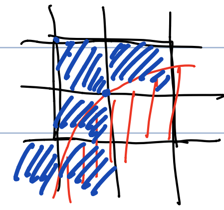
$\cdot\ c>0,\ c<\frac{1}{\sqrt{2}}$

construct grid of size $\overset{>0}{d}$.



• If K' = union of closed boxes that intersects K.

then $K' \subset \Omega$, $K \subset \mathrm{Int}(K')$.



• Consider the boundary of K', as a union of little segments, $\gamma_1, \dots, \gamma_N$, each γ_i is a side of some boxes in K', each γ_i is of length d.

$\forall z \in \mathrm{Int}(K')$

$$\cdot \quad f(z) = \frac{1}{2\pi i}\int_{\partial K'} \frac{f(\omega)}{\omega - z}\,dw = \sum_{i=1}^{N} \frac{1}{2\pi i}\int_{\gamma_i} \frac{f(\omega)}{w-z}\,dw. \qquad \square$$

Step ② : Approximate integral by Riemann sum.

take γ to be one of the γ_i

Approximate $\int_\gamma \frac{f(\omega)}{\omega - z}\,dw$ by Riemman sum.

$F(t,z) : [0,1]\times K \to \mathbb{C}$ continuous,

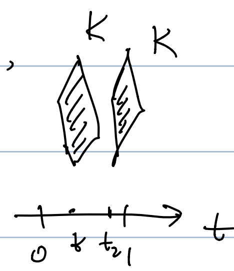
$\gamma : [0,1] \to \mathbb{C}$,

$$I = \int_0^1 \Big\{ \overbrace{\frac{f(\gamma(t))}{\gamma(t) - z} \cdot \gamma'(t)}^{F} \Big\}\,dt.$$

Then:

$\forall \varepsilon > 0$, $\exists\, \delta > 0$, such that $\forall\ t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta$,

$$\sup_{z\in K} |F(t_1, z) - F(t_2, z)| < \varepsilon.$$



choose n large enough, so that $\frac{1}{n} < \delta$,

$$I_n(z) := \sum_{i=1}^{n} F\left(\frac{i}{n}, z\right)\cdot\frac{1}{n} \longrightarrow I(z) = \int_0^1 F(t,z)\cdot dt.$$

$I_n(z) \to I(z)$ uniformly on K,

indeed, $$\sup_{z\in K} |I_n(z) - I(z)| = \sup_{z\in K}\Big| \sum_{i=1}^{n} \Big\{ F\left(\frac{i}{n}, z\right) \underbrace{\int_{\frac{i-1}{n}}^{\frac{i}{n}} dt}_{=\frac{1}{n}} - \int_{\frac{i-1}{n}}^{\frac{i}{n}} F(t,z)\,dt \Big\}\Big|$$

$$= \sup_{z\in K} \left| \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(F(\tfrac{i}{n}, z) - F(t,z) \right) dt \right| .$$

$$\le \sup_{z\in K} \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \underbrace{\left| F(\tfrac{i}{n}, z) - F(t,z) \right|}_{<\varepsilon.} dt .$$

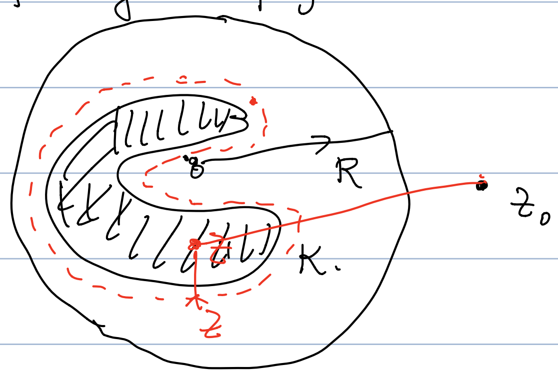
$$\le \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{i/n} \varepsilon \cdot dt = \sum_{i=1}^{n} \varepsilon \cdot \frac{1}{n} = \varepsilon \cdot \frac{1}{n} \cdot n = \varepsilon .$$

Then, $I_n(z) = \sum_{i=1}^{n} \frac{f(\gamma(t_i))}{\gamma(t_i) - z} \cdot \gamma'(t_i) \cdot \frac{1}{n}$. $\quad t_i = \frac{i}{n}$.

is a sum of rational functions. (actually, function with simple poles., i.e. $\frac{b}{z-a}$)

Together, this finish part 1 of Runge thm.

Step ③ : If K^c has no bouned component, then one can approx f by polynomials.



$\because$ K is compact, $\therefore$ $\exists R > 0$ large enough, such that $K \subset D_R(0)$

Step 3a

- if $z_0 \in \mathbb{C}$, $|z_0| > R$, then $\frac{1}{z - z_0}$ can be approximated by polynomials for $z \in K$, Pf: $\dfrac{1}{z-z_0} = -\dfrac{1}{z_0 - z} = -\dfrac{1}{z_0\left(1-\frac{z}{z_0}\right)}$

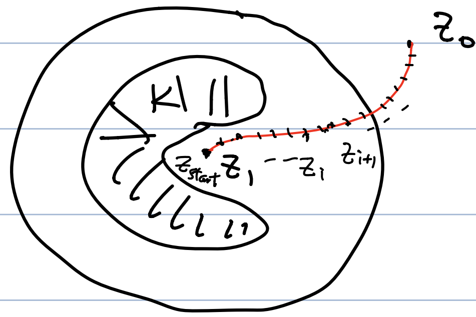
$\dfrac{|z|}{|z_0|} < \dfrac{|z|}{R} < 1.$ for $z \in K.$ $\quad = -\dfrac{1}{z_0} \displaystyle\sum_{n=0}^{\infty} \left(\frac{z}{z_0}\right)^n.$

converges uniformly for $z \in K$.

i.e. $P_N(z) = -\dfrac{1}{z_0} \displaystyle\sum_{n=0}^{N} \left(\frac{z}{z_0}\right)^n$, then

$$\left\| P_N(z) - \frac{1}{z-z_0} \right\|_K \to 0 \quad \text{as } N \to \infty.$$

step 3b: for any $z_1 \in K^c$, $\exists$ a path from z_1 to some point z_0, where $|z_0| > R$.



goal: $\dfrac{1}{z-z_i}$ can be approximated by polynomials in $\dfrac{1}{z-z_{i+1}}$

$\Rightarrow$ $\dfrac{1}{z-z_{start}}$ can be approximated by polynomial in $\dfrac{1}{z-z_{end}}$ uniformly for $z \in K$. (Read Stein for details).

Review: