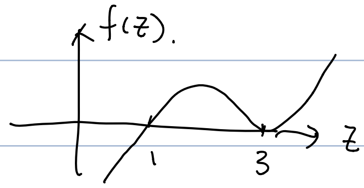
Today: Ch3 Stein. §1. §2.

- zero. pole.
- residue theorem.

Ex: • $f(z) = (z-1)(z-3)^2$ has zero at $z=1$ and $z=3$

order of zero : 1 2.



• $f(z) = \frac{1}{z-1} + \frac{1}{(z+2)^3}$ has pole at

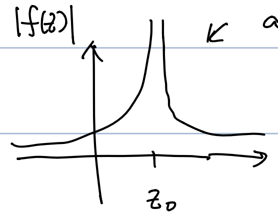
order of pole. ($\frac{1}{(z-z_0)^{\#\text{order of the pole}}}$)

$z=1$ 1

$z=-2$ 3

• $f(z) = \frac{1}{(z-1)(z+2)^3}$ has the same pole structure as pole.

"pole" :



a high "pole"

$\frac{1}{z-z_0}$

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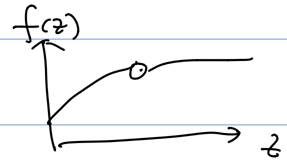
⊖ overview for this chapter :

- classify zero & pole.
- residue thm : residue of a function at a point z_0 is the coeff $f(z) = \dots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0)^1 \dots$

a_{-1}

- classification of isolated singularities (where value of the function is undefined.).

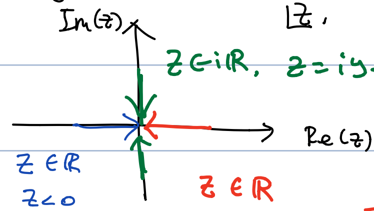


- removable singularity.

- pole : near z_0, $f(z) \sim \dfrac{g(z)}{(z-z_0)^n}$ $\quad g(z)$: hol'c near z_0

- <u>essential</u> singularity : e.g. $e^{\frac{1}{z}}$. near 0.

$e^{\frac{1}{z}}$



$|z|$,

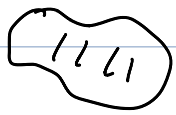
$z \in i\mathbb{R}$, $z = iy$. $y > 0$. $e^{\frac{1}{iy}}$ oscillate. $\left|e^{\frac{1}{iy}}\right| = 1$

$z \in \mathbb{R}$, $z < 0$, $e^{\frac{1}{z}} \to 0$

$z \in \mathbb{R}$, $z \to 0^+$, $e^{\frac{1}{z}} \to +\infty$

- Harmonic functions. Maximum Principle.
  - Ex: $\mathrm{Re}(f(z))$, $\mathrm{Im}(f(z))$ ~~is~~ are harmonic function,

    if $|f(z)| \neq 0$, $\log|f(z)|$ is also harmonic.

  -  Ω maximum principle says:

    harmonic function takes its max only at the boundary of Ω.

    (e.g. $u(x,y) = x^2 - y^2$. or $u(x) = x$).

- <u>Today</u>: §1: Say $f: \Omega \to \mathbb{C}$ is holomorphic function. and for $z_0 \in \Omega$, $f(z_0) = 0$.

  <u>Q:</u> how to define the order of zero at z_0?

  <u>Ans:</u> Do Taylor expansion at z_0.

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z-z_0)^n. = \overset{\neq 0}{a_m} \cdot (z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \cdots$$

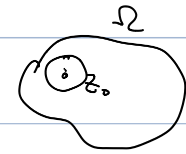
look the first non-zero a_n. $\underline{m}$ is the order of zero.

$$a_m = \frac{f^{(m)}(z_0)}{m!}$$

(1.1)

Thm: Suppose f is holo'c in a open connected set Ω, f has zero z_0 in Ω, and f doesn't vanish identically in Ω. Then there exist a nbhd $U \subset \Omega$, of z_0, and a non-vanishing hol'c function g and a unique positive integer n, s.t.

$$f(z) = (z-z_0)^n \cdot \underline{g(z)}. \qquad \forall z \in U.$$



Sketch of the proof:

① Do Taylor expansion at z_0.

$$f(z) = \underset{\substack{\uparrow \\ a_n \neq 0.}}{a_n} \cdot (z-z_0)^n + a_{n+1} \cdot (z-z_0)^{n+1} + \cdots$$

valid for $|z-z_0| < r$

$$= (z-z_0)^n \cdot \underbrace{(a_n + a_{n+1}(z-z_0)^1 + \cdots)}.$$

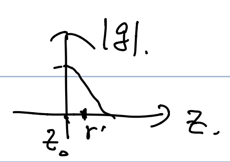
$$= (z-z_0)^n \, g(z).$$

$g(z): D_r(z_0) \to \mathbb{C}$ hol'c.

② Shrink the radius r to r', s.t.

$g(z)$ is non-vanishing on $\underset{\substack{\shortparallel \\ U}}{D_{r'}(z_0)}$.

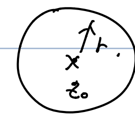
This is always possible, since $|g(z_0)| = |a_n| > 0$ and $|g|: D_r(z_0) \to \mathbb{R}$ is a continuous function.



- pole : • Let $z_0 \in \mathbb{C}$, a deleted nbhd of z_0.

is an open disk, centered at z_0, but removing the pt z_0.

$$D_r^{\times}(z_0) = \{ z \mid 0 < |z - z_0| < r \}$$



- $f(z)$ has an isolated singularity at z_0, iff $\exists$ a deleted nbhd $D_r^{\times}(z_0)$, s.t. f is hol'c on $D_r^{\times}(z_0)$.

<u>Def'n</u>: $f(z)$ has a pole at z_0, if $\exists\, D_r^{\times}(z_0)$, s.t.

① f is hol'c and non-vanishing on $D_r^{\times}(z_0)$

② and the function

$$\frac{1}{f(z)} = F(z) = \begin{cases} 0 & z = z_0 \\ \dfrac{1}{f(z)} & z \in D_r^{\times}(z_0). \end{cases}$$

is $\boxed{\text{hol'c}}$ on $D_r(z_0)$.

<u>Ex</u>: $f = e^{\frac{1}{z}}$ on $D_1(z_0 = 0)$ is hol'c and non-vanishing. on $D^{\times}(z=0)$, but $\frac{1}{f}$ cannot be defined on $D_1(z_0)$

<u>Thm (1.2)</u>: If f has a pole at $z_0 \in \Omega$, then, in a nbhd of z_0, there is a hol'c non-vanishing function h. and a unique positive integer n, such that.

$$f(z) = \frac{h(z)}{(z - z_0)^n}.$$

<u>Pf</u>: Consider the function $\frac{1}{f(z)}$, it is hol'c near z_0. and has a zero at z_0, Apply the previous thm.

$$\frac{1}{f(z)} = (z - z_0)^n \cdot g(z), \quad g(z) \neq 0 \quad \text{for } z \in D_r(z_0)$$

$$\Rightarrow \quad f(z) = \frac{1}{(z - z_0)^n} \cdot \frac{1}{g(z)} \quad \forall\, z \in D_r^{\times}(z_0) \qquad \#.$$

$\frac{1}{g(z)} = h(z)$, $h(z) \neq 0 \quad \forall\, z \in D_r(z_0)$

Def: n is the order of the pole at z_0.

Thm (1.3): If $f(z)$ has a pole of order n at z_0, then near z_0, we have

blow up fastest

$$f(z) = \underbrace{\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z-z_0)}}_{\text{"principal part" of } f \text{ at } z_0} + \underbrace{G(z)}_{\text{"regular part"}}\ldots$$

where $G(z)$ is hol'c near z_0, $a_{-n} \neq 0$.

Pf: take $f(z) = \dfrac{h(z)}{(z-z_0)^n}$, Taylor expand $h(z)$ at z_0.

Def: the coeff a_{-1} in the above expansion ($= \mathrm{res}_{z_0} f$) to be residue of f at z_0.

- $$\int_{C_\varepsilon(z_0)} f(z)\cdot dz = (2\pi i)\cdot(\mathrm{res}_{z_0} f).$$ ...... baby form of residue theorem.

Ex: $f(z) = \dfrac{e^z}{z-1}$ has a pole at $z=1$.

$\mathrm{Res}_1 f = ?$

$$f(z) = \frac{\text{Taylor series of } e^z \text{ at } z=1}{z-1}$$

$$e^z = e^{(z-1)} \cdot e^1$$

$$= e^1 \cdot \left(1 + (z-1) + \frac{(z-1)^2}{2!} + \cdots \right)$$

$$f(z) = \frac{e \cdot \left(1 + (z-1) + \frac{(z-1)^2}{2!} + \cdots \right)}{(z-1)}$$

$$= \frac{\overset{a_{-1}}{e}}{z-1} + \underbrace{e + \frac{e \cdot (z-1)}{2!} + \cdots}_{G(z)\ :\ \text{regular part.}}$$

$$\operatorname{Res}_{z=1} f = e.$$

In general, if $f(z) = \dfrac{h(z)}{z - z_0}$ $h(z)$ non-vanishing near z_0,

then

$$\boxed{\operatorname{Res}_{z_0} f = h(z_0).}$$

(*) holds if f has a simple pole, or pole of order 1.

- Ex 2 : $f(z) = \dfrac{1}{(z-1)(z-2)}$

$$\operatorname{Res}_{z=1} f = h(1) = \frac{1}{1-2} = -1$$

$f(z) = \dfrac{h(z)}{(z-1)}$, $h(z) = \dfrac{1}{z-2}$

($h(z)$ is well defined and non-vanishing near $z=1$.)

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$$\operatorname{Res}_{z=2} f = g(1) = \frac{1}{2-1} = 1.$$

near $z=2$, $f(z) = \dfrac{g(z)}{z-2}$

where $g(z) = \dfrac{1}{z-1}$. well-defined, non-vanishing near $z=2$.

(Ex fun : $f(z) = \dfrac{1}{(z-a)(z-b)(z-c)}$.

check : $\operatorname{Res}_a f + \operatorname{Res}_b f + \operatorname{Res}_c f = 0.$).

In general:

Thm: if f has an order n pole at z_0, i.e.

$$f(z) = \frac{h(z)}{(z-z_0)^n} \quad \text{near } z_0.$$

and if $h(z) = h(z_0) + \frac{h'(z_0)}{1!}\cdot(z-z_0) + \cdots + \frac{h^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1} + \cdots$

then $\operatorname{Res}_{z_0} f = \frac{h^{(n-1)}(z_0)}{(n-1)!} = \lim_{z\to z_0} \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1} \underbrace{(z-z_0)^n f(z)}_{h(z).}$

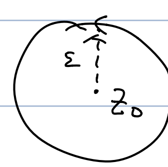
Pf: plug in the Taylor expansion of $h(z)$.

Ex: $f(z) = \frac{e^z}{(z-5)^2}$ let $h(z) = e^z$, $n=2$. $z_0 = 5$

$$h'(z_0) = (e^z)'\Big|_{z=5} = e^z\Big|_{z=5} = e^5.$$

$$\operatorname{Res}_5 f(z) = \frac{h'(z_0)}{(2-1)!} = \frac{e^5}{1} = e^5.$$

Baby form of Residue Thm:



$$\int_{C_\varepsilon(z_0)} f(z)\, dz$$

Expand $f(z)$ in $D_\varepsilon(z_0)$ as

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{(z-z_0)} + G(z).$$

$$= \int_{C_\varepsilon(z_0)} \frac{a_{-n}}{(z-z_0)^n}\, dz + \cdots + \int \frac{a_{-2}}{(z-z_0)^2}\, dz$$

$$+ \oint_{C_{\varepsilon}(z_0)} \frac{a_{-1}}{z-z_0}\, dz + \underbrace{\oint_{C_{\varepsilon}(z_0)} G(z)\, dz}_{= 0 \text{ by Cauchy Thm.}}$$

define $u = z - z_0$. $du = dz$

$$\oint_{C_{\varepsilon_0}(z_0)} \frac{1}{(z-z_0)^k}\, dz = \oint_{|u|=\varepsilon} \frac{1}{u^k}\, du. \qquad u = e^{i\theta} \cdot \varepsilon$$

$$= \int_{\theta=0}^{2\pi} \frac{1}{(\varepsilon e^{i\theta})^k}\, \varepsilon \cdot e^{i\theta} \cdot i \cdot d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{i}{\varepsilon^{k-1} \cdot e^{i\theta(k-1)}}\, d\theta$$

$$= \begin{cases} i \cdot 2\pi & k = 1 \\ 0 & k \neq 1 \end{cases}$$

$$= a_{-1} \cdot (2\pi i)$$