· Today : ① Residue Theorem

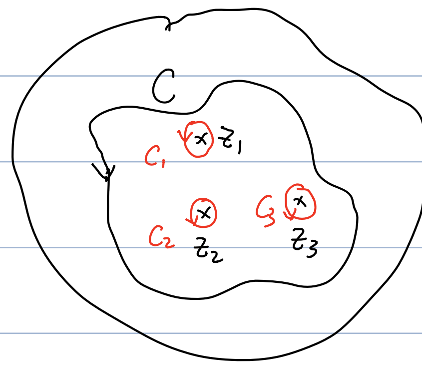
② index of a curve around a point ("winding number of a curve around a pt").

③ Example of applications for residue thm. (Stein. Ch 3. §2. Ahlfors. Ch 4.)

① Residue Thm: Let $f: \Omega \to \mathbb{C}$ be a $\underbrace{\text{hol'c function}}_{\text{except at poles } z_1, \cdots, z_k.}$ on an open set Ω, and let C be a simple closed curve in Ω, such that the interior of C is also contain in Ω, and $z_1, \cdots, z_k$ are inside C. Then

$$\int_C f(z)\, dz = 2\pi i \cdot \sum_{i=1}^{k} \mathrm{Res}_{z_i} f$$

Ω



Pf: deform C so that C breaks into small circles around each z_i, say $C_\varepsilon(z_i) \overset{=}{} C_i$. Then, use.

$$\int_{C_\varepsilon(z_i)} f(z)\, dz \underset{(*)}{=} 2\pi i \cdot \mathrm{Res}_{z_i} f$$

Indeed, $(*)$ holds by expanding $f(z)$ near z_i into form

$$f(z) = \frac{a_{-n_i}}{(z-z_i)^{n_i}} + \frac{a_{-n_i+1}}{(z-z_i)^{n_i-1}} + \cdots + \frac{a_{-1}}{(z-z_i)^1} + G(z).$$

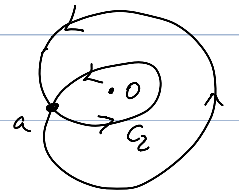
$$\int_{C_\varepsilon(z_i)} f(z)\, dz = \int_{C_\varepsilon(z_i)} \frac{a_{-1}}{z-z_i}\, dz = 2\pi i \cdot a_{-1} = 2\pi i \cdot \mathrm{Res}_{z_i} f.$$

#.

Rmk: The limitation on the curve C to be "simple".

can be remove, i.e. we can consider crossings. i

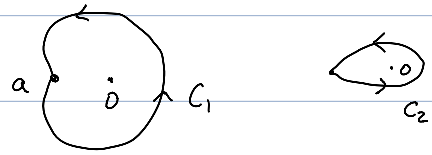
C:



$$\int_C \frac{1}{z}\,dz = 2\pi i \cdot \underline{\underline{2}}.$$

"winding number of C around 0".

$C = C_1 + C_2$



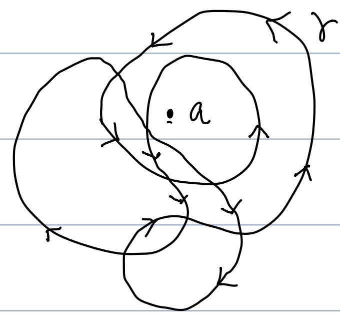
integration along C = int along C_1 then int along C_2

P115.
(Ahlfors)

Lemma: If a (piecewise smooth) closed curve γ does not pass through a point $\underline{\underline{a}}$, then the value of the integral

$$\int_\gamma \frac{1}{z-a}\,dz$$

is a multiple of $2\pi i$.



First observation: $\frac{1}{z-a}\,dz = \frac{1}{z-a}\,d(z-a) = d\cdot \log(z-a)$

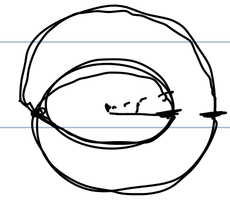
$$\log(z-a) = \operatorname{Re}(\log(z-a)) + i\operatorname{Im}(\log(z-a)).$$
$$= \underbrace{\log|z-a|}_{\text{well-defined for } |z-a|>0.} + \underbrace{i\cdot \arg(z-a)}_{\text{is only well-defined upto a multiple of } 2\pi i.}.$$

informally, $\int_\gamma \frac{1}{z-a}\,dz = i\cdot$("change of the argument" of $(z-a)$)

check: $\int_{\gamma} \frac{1}{z-a}\,dz = 2\pi i$.

γ: 

$$\int_{\gamma} \frac{1}{z-a}\, dz = 4\pi i$$



Formal proof: If γ is parametrized by $z(t)$. $\alpha \le t \le \beta$. then, we can consider the function

$$h(t) = \int_{\alpha}^{t} \frac{z'(s)}{z(s)-a}\, ds.$$

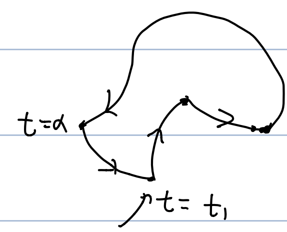
$\alpha \quad t \quad \beta$

$$h(\beta) = \int_{\gamma} \frac{1}{z-a}\, dz.$$

It is defined and is continuous on the closed interval $[\alpha, \beta]$,

$$h'(t) = \frac{z'(t)}{z(t)-a}$$

whenever $z'(t)$ is continuous.



$z'(t)$ is not continuous at $t=t_1$

Then, the combination $\underbrace{e^{-h(t)} \cdot (z(t)-a)}_{H(t)}$ has derivative vanishes every where in $t \in [\alpha, \beta]$, except at possibly finite many points. This function $H(t)$ is continuous, hence.

$H(t) =$ const along $t \in [\alpha, \beta]$.

Thus. $$e^{-h(t)}(z(t)-a) = e^{-h(\alpha)}(z(\alpha)-a)$$
$$= z(\alpha) - a.$$

$$e^{h(t)} = \frac{z(t)-a}{z(\alpha)-a}.$$

$$e^{h(\beta)} = \frac{z(\beta)-a}{z(\alpha)-a} = 1 \qquad \therefore \quad h(\beta) = 2\pi i \cdot \underline{\underline{n}}.$$

<u>Def</u>: $$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz$$ winding number of γ around a.

<u>Rmk</u>:

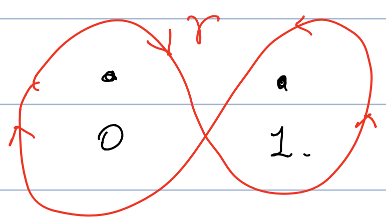
- $n(-\gamma, a) = -n(\gamma, a)$.

<u>Slightly more general version of residue thm</u>:

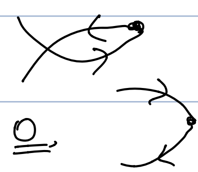
- $f : \Omega \to \mathbb{C}$ has finitely many poles $z_1, \cdots, z_k$.
- γ is a curve in Ω, avoiding these poles.

$$\int_\gamma f(z)\,dz = 2\pi i \cdot \sum_{i=1}^{k} \left(\mathrm{Res}_{z_i} f\right) \cdot \underline{\underline{n(\gamma, z_i)}}$$

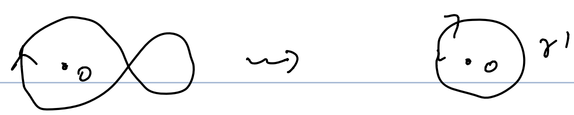
<u>Ex</u>:



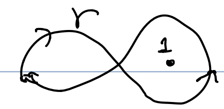
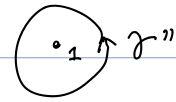
$$f(z) = \frac{\alpha}{z} + \frac{\beta}{z-1}$$



$n(\gamma, 0) =$ we can <u>deform</u> γ, as long as γ avoid $\underline{\underline{0}}$.



$= (-1)$ minus because γ' winds clockwise.

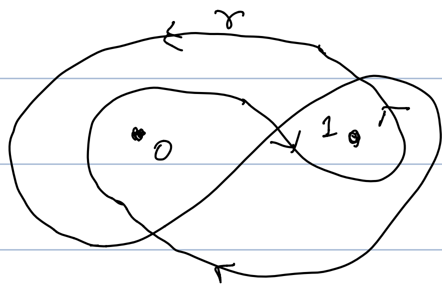
$n(\gamma, 1) =$  $\rightsquigarrow$ 

$= 1.$ $\qquad n(\gamma'', 1) = 1.$

$$\int_\gamma f(z)\,dz = 2\pi i \cdot \Big(\mathrm{Res}_0 f \cdot n(\gamma, 0) + \mathrm{Res}_1 f \cdot n(\gamma, 1) \Big)$$

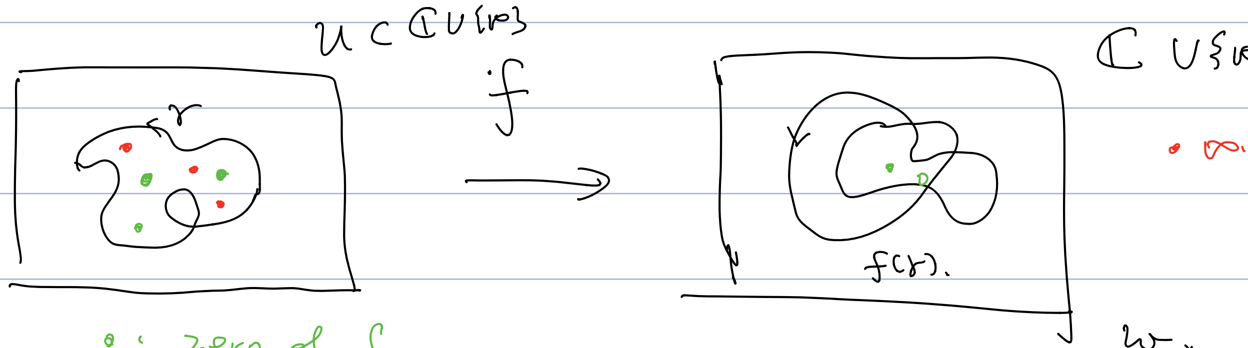
$$= 2\pi i \cdot \big(\alpha \cdot (-1) + \beta(+1) \big)$$

$$= 2\pi i \cdot (\beta - \alpha).$$

Ex : 

$n(\gamma, 0) = 0$

$n(\gamma, 1) = 0.$



∘ : zero of f

∘ : ∞ of f

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_{f(\gamma)} \frac{1}{w}\,dw$$

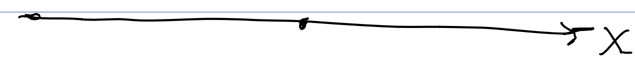
$w = f(z).$

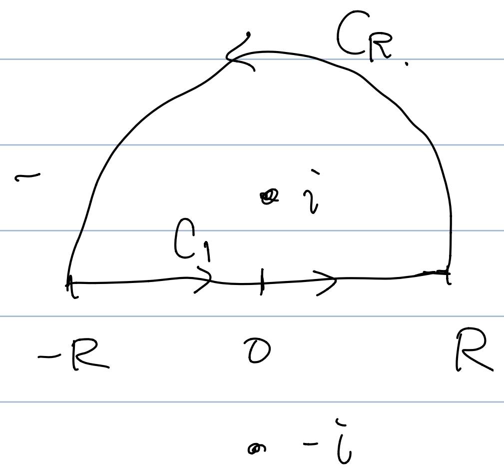
(argument principle next time).

- Application of Residue Thm.

(evaluate definite real integral).

Ex
$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{R\to+\infty} \underbrace{\int_{-R}^{R} \frac{1}{1+x^2}\,dx}_{I_R}.$$





$f(z) = \frac{1}{1+z^2}$

$f(z)$ has poles at roots of z^2+1, i.e.,

$(z+i)(z-i) = 0$

pole at $z=i$, $z=-i$

$C = C_1 + C_R$.

$$\int_C f(z)\,dz = 2\pi i \cdot \operatorname{Res}_{z=i} f(z)$$

order 1 pole at $f(z)$.

$$\operatorname{Res}_{z=i} f(z) = \lim_{z\to i} f(z)\cdot(z-i)$$

$$= \lim_{z-i} \frac{(z-i)}{(z-i)(z+i)}$$

$$= \left.\frac{1}{z+i}\right|_{z=i} = \frac{1}{2i}$$

$$\int_C f(z)\,dz = 2\pi i \cdot \frac{1}{2i} = \pi.$$

<u>claim:</u>

$$\left| \int_{C_R} \frac{1}{z^2+1}\,dz \right| \le \sup_{C_R} f(z) \cdot |C_R|$$

$$\le \frac{C}{R^2} \cdot 2\pi R \to 0$$

as $R \to \infty$.

Thus:
$$\lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz = \int_C f(z)\,dz - \underbrace{\lim_{R\to\infty}\int_{C_R} f(z)\,dz}_{\to 0}$$

$$= \int_C f(z)\,dz = \pi.$$

- <u>Rmk</u>: for $f(x) = \frac{P(x)}{Q(x)}$, such that $Q(x)$ has no roots along the real line. and $\deg P(x) \le \deg Q(x) - 2$ (proof as above) then. $\int_{-\infty}^{+\infty} f(x)\,dx$ can be evaluated in the same way,

- <u>Rmk (quiz)</u>: can we close up the contour from below, like.  (yes!)

Ex: $I = \int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^x}\, dx \qquad 0 < a < 1$

First check: is this well defined?

Near $+\infty$. $\sim \int_R^{\infty} \frac{e^{ax}}{e^x}\, dx \sim \int_R^{\infty} \underline{e^{(a-1)x}}\, dx < \infty$. $\quad a-1<0$

near $-\infty$ $\sim \int_{-\infty}^{-R} \frac{e^{ax}}{1}\, dx \sim -\int_R^{\infty} \underline{e^{-au}}\, du < \infty$

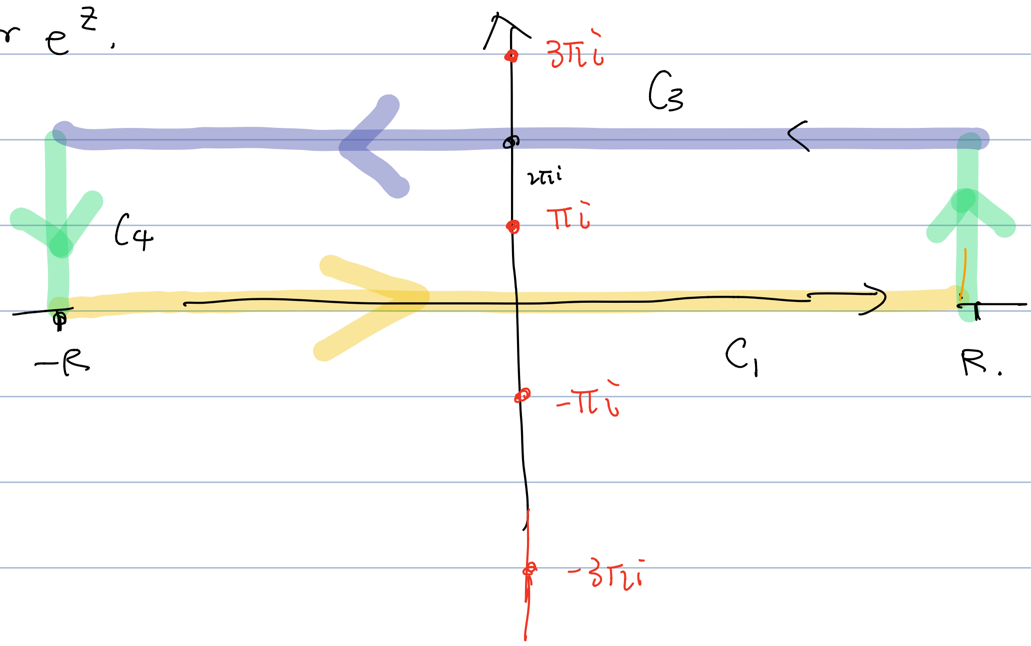
$u=-x$ again exp decay

✓

- what are the zeros of $e^z+1=0$?

$e^{\pi i} = -1$. $\therefore e^{\pi i}+1=0$

but $z = \pi i + 2\pi i \cdot n \quad n \in \mathbb{Z}$ all have same value for e^z.



want: $\underbrace{\int_{C_1} f(z)\, dz}_{= I_1}$.

know $I_C = \int_C f(z) = 2\pi i \cdot \text{Res}_{z=\pi i}(f(z))$

$= 2\pi i \cdot (-e^{a\pi i})$.

if $\alpha(z)$ continuous at z_0.

$$\lim_{z\to z_0} \alpha(z)\cdot\beta(z) = \alpha(z_0) \lim_{z\to z_0} \beta(z)$$

$$\operatorname{Res}_{\pi i} f(z) = \lim_{z\to\pi i} (z-\pi i)\cdot f(z).$$

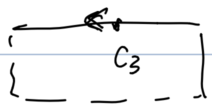
or apply L'hopital.

$$= \lim_{z\to\pi i} \frac{(e^{az})}{1+e^{z}} \cdot (z-\pi i) = \frac{e^{a\cdot\pi i}}{\lim\limits_{z\to\pi i} \frac{1+e^{z}}{z-\pi i}} = \frac{e^{a\pi i}}{-1}$$

$$\lim_{z\to\pi i} \frac{1+e^{z}}{z-\pi i} = (1+e^{z})'\Big|_{z=\pi i} = e^{z}\Big|_{z=\pi i} = -1.$$

$$\int_{C_3} f(z)\, dz = \int_{u=+\infty}^{-\infty} \frac{e^{a(2\pi i+u)}}{1+e^{(2\pi i+u)}} d(2\pi i+u) = -\int_{-\infty}^{+\infty} \frac{e^{a\cdot 2\pi i}\cdot e^{au}}{1+e^{u}}\, du$$

$z = 2\pi i + \underline{u}$ $\quad$ u goes from $+\infty$ to $-\infty$



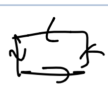
$$= -e^{a\cdot 2\pi i} \int_{-\infty}^{+\infty} \frac{e^{au}}{1+e^{u}}\, du$$

$$= -e^{a\cdot 2\pi i}\cdot I_1$$

- one can also show along vertical edges.

$$I_2 = \int_{C_2} f(z)\, dz \longrightarrow 0 \quad \text{as } R\to\infty$$

$$I_4 = \int_{C_4} f\cdot dz \to 0.$$



$$\therefore\ I_C = I_1 + I_2 + I_3 + I_4.$$

$$\xrightarrow{R\to\infty} \lim_{R\to\infty} I_1 + I_3 = \lim_{R\to\infty} (1-e^{a\cdot 2\pi i})\, I_1.$$

$$= (1-e^{a\cdot 2\pi i})\cdot I$$

$$I = \frac{I_c}{1-e^{2\pi i\cdot a}} = \frac{2\pi i\cdot\left(-e^{a\pi i}\right)}{1-e^{2\pi i\cdot a}}$$

$$= \frac{2\pi i}{e^{\pi i a} - e^{-\pi a i}} = \frac{\pi}{\dfrac{e^{\pi i a} - e^{-\pi i a}}{2i}}$$

$$= \frac{\pi}{\sin(\pi a)} > 0. \qquad \because 0<a<1$$

$$\therefore \sin(\pi a) > 0.$$