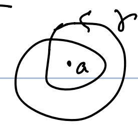
- Midterm 2 date: possibly 2nd week of Nov.

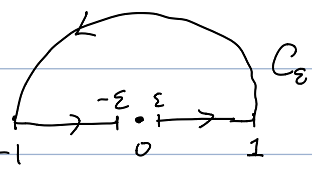
  after we finish Ch 3 in Stein ($\approx$ Ch 4 of Ahlfors).

Q: is it possible to have fractional winding number?

- recall the $n(\gamma, a) = \frac{1}{2\pi i}\oint \frac{1}{z-a}\,dz$



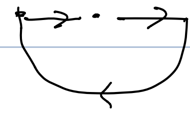
  where curve γ does not pass through a.



Ex:

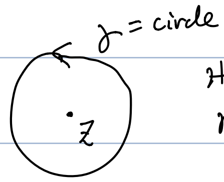
$$I = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{C_\varepsilon} \frac{1}{z}\,dz =: \frac{1}{2\pi i}\int \frac{1}{z}\,dz.$$

if $C =$



$$\frac{1}{2\pi i}\int_C \frac{1}{z}\,dz = -\frac{1}{2}$$

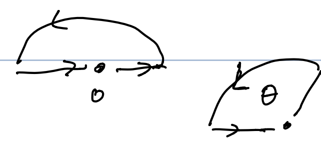
it turns out $I = \frac{1}{2}$.



How does $n(\gamma, z)$ depends on z as z moves around?

- For z inside γ, $n(\gamma, z) = 1$
- For z outside γ, $n(\gamma, z) = 0$,
- so heuristically, for z on γ, we have

  $n(\gamma, z) = \frac{1}{2}$.



Challenge question: can $n(\gamma, z)$ be other frations?, $\frac{1}{m}$?

§3 Ch 3. Stein. Classification of singularity and meromorphic function.

Types of singularities:

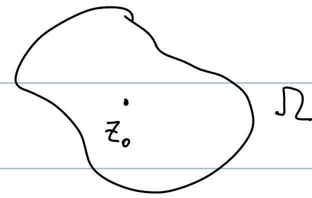
- removable singularity.
- pole. : pole at $z_0 \Leftrightarrow (z-z_0)^m f(z)$

- essential singularity
  (all other kinds)

is holomorphic for m large enough.

Thm 3.1 (Removable Singularity).

Suppose f is hol'c in an open set Ω, except possibly at a point $z_0 \in \Omega$. If f is bounded on $\Omega - \{z_0\}$ then z_0 is a removable singularity.



Observations: · if f can be extended to z_0, (hol'c)

what value of f has to be at z_0?

- Since f is continuous, then $f(z_0) = \lim_{z \to z_0} f(z)$,

  But at this moment, without knowing f is hol'c at z_0. $\lim_{z \to z_0} f(z)$ might not exist. i.e. has has different value as $z \to z_0$.

- If f can be extended holomorphically to z_0, then

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_\varepsilon(z_0)} \frac{f(w)}{w - z_0}\, dw.$$ has to be true.

Can we declare that,

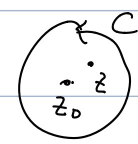
① put back the value $\frac{1}{2\pi i} \oint_{C_\varepsilon(z_0)} \frac{f(w)}{w - z_0}\, dw$, " $I(z_0)$ "

then declare done?

Problem: $I(z_0)$ may not be $\lim_{z \to z_0} f(z)$.

Pf: Fix a small circle $C = C_\varepsilon(z_0)$ around z_0, so that $\overline{D_\varepsilon(z_0)} \subset \Omega$.

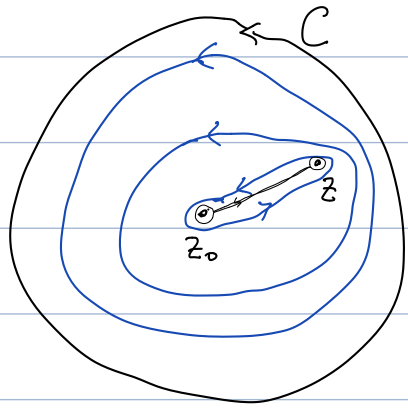
Define a function in $D_\varepsilon(z_0)$



$$g(z) := \frac{1}{2\pi i} \oint \frac{f(w)}{w - z}\, dw$$

Now, suffice to prove that $g(z) = f(z)$, $\forall z \in D_\varepsilon^\times(z_0)$

Deform C to be 2 circles, one around z_0, one around z.



$C_\delta(z_0)$ and $C_\delta(z)$

$$g(z) = \frac{1}{2\pi i}\int_{C_\delta(z_0)} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_{C_\delta(z)} \frac{f(w)}{w-z}\,dw.$$

<u>Claim</u>,

$$\xrightarrow{\delta \to 0.} \quad 0 \quad + \quad f(z).$$

$$\left|\int_{C_\delta(z_0)} \underset{\substack{\uparrow \\ \text{bounded function} \\ \text{in } D_\delta(z_0)}}{\frac{f(w)}{w-z}}\,dw\right| \le \underbrace{\sup_{w\in C_\delta(z_0)} \left|\frac{f(w)}{w-z}\right|} \cdot (2\pi\delta) \xrightarrow{\text{as } \delta\to 0.} 0$$

#.

<u>Cor</u> : $f(z)$ has a pole at z_0

$\Leftrightarrow$ $\underline{|f(z)| \to \infty}$ as $z \to z_0$.

<u>Pf</u> : $\Rightarrow$ By definition, $f(z)$ has a pole at z_0 means, "$\exists\, F(z)$ hol'c at nbhd of z_0, $F(z) = \frac{1}{f(z)}$ in $D_\varepsilon^\times(z_0)$,

and $F(z_0)=0$ " $\Rightarrow \frac{1}{|f(z)|} \to 0$, as $z \to z_0$, hence.

$$\frac{1}{|f(z)|} \to 0 \text{ as } z \to z_0, \Leftrightarrow |f(z)| \to \infty \text{ as } z \to z_0.$$

$$|f(z)| \to \infty \qquad \frac{1}{f(z)} \to 0$$

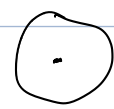
$$\Leftarrow \text{ as } z \to z_0, \Leftrightarrow \frac{1}{|f(z)|} \to 0, \text{ in particular.}$$

$\frac{1}{f(z)}$ is bounded in a deleted nbhd of z_0.

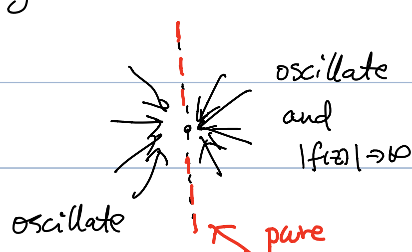
Hence, by the thm, we know $\frac{1}{f(z)}$ can be extended to z_0, and the value of extension of $\frac{1}{f}$ at z_0 has to be zero, by continuity. #

<u>Observation</u>: this corollary says that, if $\underline{|f(z)| \to \infty}$ as $z \to z_0$, then z_0 cannot be an essential singularity.

Consider: $f(z) = e^{\frac{1}{z}}$ near $z=0$,



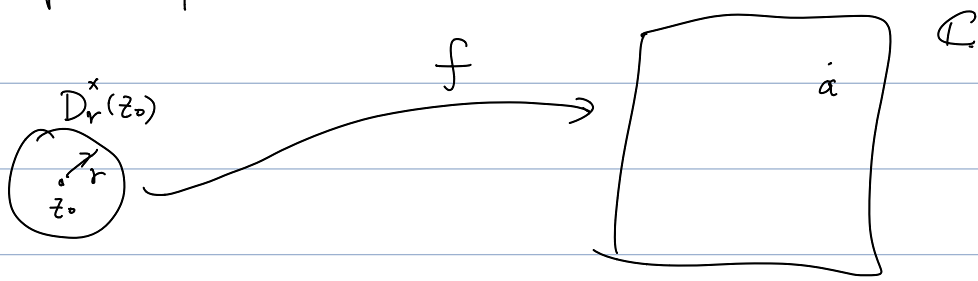
- if $z \to 0$ along positive real direction, $z=r$. $(r>0)$
  then $e^{\frac{1}{r}} \to \infty$
- if $z \to 0$ along negative real direction, $z=-r$.
  $e^{-\frac{1}{r}}$ "$\approx e^{-\infty}$" $\to 0$ as $r \to 0$
- if $z \to 0$ along other directions, you see oscillation together with decay or blow up.



and $|f(z)| \to 0$ oscillation along imaginary axis.

Thm (Casorati - Weierstrass Thm).

If f is hol'c in a punctured disk $D_r^{\times}(z_0) = D_r(z_0) - \{z_0\}$, and has essential singularity at z_0, then. the image of $D_r^{\times}(z_0)$ under f is dense in the complex plane $\mathbb{C}$.



i.e. dense means $\forall\, w \in \mathbb{C}$, $\forall\, \delta > 0$, $D_\delta(w) \cap f(D_r^{\times}(z_0)) \neq \phi$.

Pf: We prove by contradiction. Suppose, $\exists\, w \in \mathbb{C}$, and $\delta > 0$, s.t. $D_\delta(w) \cap f^{\times}(D_r^{\times}(z_0)) = \phi$. Then

Consider

$$g(z) = \frac{1}{f(z) - w} \quad \text{for } z \in D_r^{\times}(z_0)$$

$\because\ |f(z) - w| \geq \delta \quad \therefore\ |g(z)| \leq \underline{\frac{1}{\delta}}, \ \forall\, z \in D_r^{\times}(z_0)$

by the "Riemann removable singularity thm".

Hence $g(z)$ can be extend to $D_r(z_0)$.

- If $g(z_0) \neq 0$, then $\frac{1}{g(z)} + w$ is an extension of f over z_0. i.e. z_0 is a removable singularity.

- If $g(z_0) = 0$, then $f(z) - w$ has a pole at z_0.

Either way, it contradicts with "f has an essential singularity at z_0".

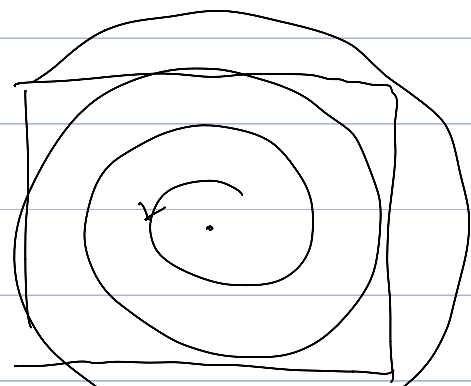
Ex (cont.) $\quad z = r \cdot e^{i\theta}$, $\quad r \to 0$

then

$$e^{\frac{1}{z}} = e^{\frac{1}{r \cdot e^{i\theta}}} = e^{\frac{1}{r} \cdot e^{-i\theta}}$$

$$= e^{\frac{1}{r}(\cos\theta - i\sin\theta)}$$

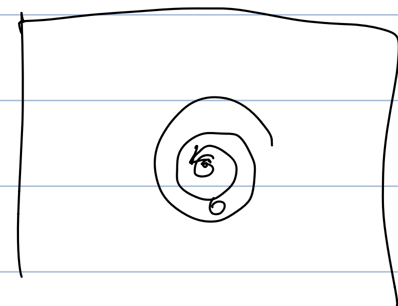
$$= e^{\frac{1}{r}\cos\theta} \cdot e^{-i\frac{1}{r}\cdot\sin\theta}$$



For $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

as $r \to 0$.

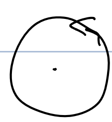
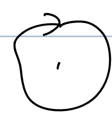
- for different θ, the spiralling rate is different.



For $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$,

spiralling inward.

For $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, $\quad \sin\theta = +1$ or -1, so.

one gets  or. 

#

- Meromorphic Function, Riemann sphere " a.k.a. extended complex place $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$".
rational function.

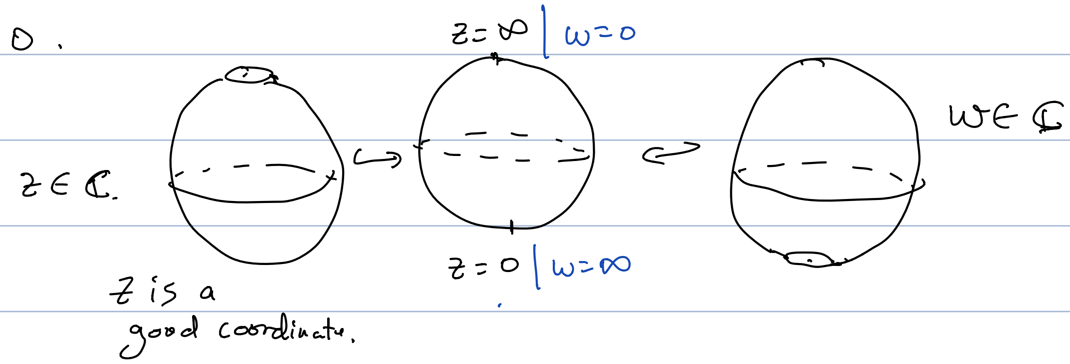
Def: A function f on an open set Ω is meromorphic, if there exists a sequence of points $\{z_0, z_1, z_2, \cdots\}$ in Ω, such that it has no limit point in Ω.

(i.e. $\forall w \in \Omega$, $\exists \delta > 0$, $D_\delta(w)$, s.t. $D_\delta(w)$ only contains 1 or 0 pt from the sequence.), and

- f is hol'c in $\Omega - \{z_0, z_1, \cdots\}$
- f has poles at $\{z_0, \cdots, z_k, \cdots\}$.

(Ex.: $\Omega = \mathbb{D}$, $f(z) = 1/\sin\left(\frac{1}{z-1}\right)$ pole at $\frac{1}{z-1} = n\pi$. i.e. $z = 1 + \frac{1}{n\pi}$. $\forall n \in \mathbb{Z}$)
$\Omega = \mathbb{C}$, $f(z) = 1/\sin z$.

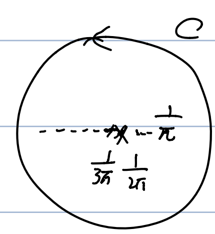
- <u>Extend complex plane</u>: $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
  - consider nbhd around $z = \infty$, by introducing new complex coord $w = 1/z$. then $z \to \infty$ corresponds to $w \to 0$.



- <u>Thm</u>: If f is a meromorphic function on the extended complex plane $\hat{\mathbb{C}}$, then f is a rational function, i.e. $f(z) = \frac{P(z)}{Q(z)}$.

Q: what would go wrong, if the poles accumulate to a point $z_0 \in \Omega$.

$$\int_{|z|=1} \frac{1}{\sin(1/z)}\, dz$$



↑ has pole at $z = \frac{1}{n\pi}$

- check: integral is well defined, $\frac{1}{\sin(1/z)}$ is finite on $\{|z|=1\}$.
- However, one cannot apply Cauchy integral formula.

<u>HW</u>:

#1: let $z = e^{i\theta}$, and integrate along $|z|=1$.

on $|z|=1$, $z \cdot \bar{z} = 1$

$$d\theta = \underbrace{\cdots\ dz}$$

$\bar{z} = 1/z$

$$\cos\theta = \mathrm{Re}(z) = \frac{z+\bar{z}}{2} = \frac{z+1/z}{2},$$

#3: $$\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+a^2}\,dx = \ ?$$

trick: $\cos z = \frac{e^{iz}+e^{-iz}}{2}$

$$\int_{-\infty}^{+\infty} \frac{e^{-ix}}{x^2+a^2}\,dx$$

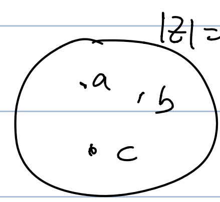
- as $|z| \to \infty$ in the upper half plane, $|e^{iz}| \to 0$, $|e^{-iz}| \to \infty$

$$\cos x = \mathrm{Re}(e^{ix}).$$

$$\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+a^2}\,dx = \int_{-\infty}^{+\infty} \mathrm{Re}\left(\frac{e^{ix}}{x^2+a^2}\,dx\right)$$

$$= \mathrm{Re}\,\underbrace{\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+a^2}\,dx}.$$

$$\oint \frac{1}{(z-a)(z-b)(z-c)}\,dz \overset{?}{=} 0$$



$z=\infty$

$$\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}$$

$$= \frac{(b-c)-(a-c)+(a-b)}{(a-b)(a-c)(b-c)} = \frac{0}{\cdots}$$

$$= 0.$$