1. Open sets and Closed sets :

- $X = \mathbb{C}$
- basic open set : $D_\varepsilon(x) = \{ y \in \mathbb{C} \mid |y-x| < \varepsilon \}$.
- $\Omega \subset \mathbb{C}$: Ω is open, if $\forall x \in \Omega$, $\exists D_\varepsilon(x)$, such $x \in D_\varepsilon(x) \subset \Omega$

- $A \subset \mathbb{C}$; A is closed if $A^c = \mathbb{C} \setminus A$ is open.

basic properties :

- $\bigcup_\alpha U_\alpha$ is open if each U_α is open.
- $\bigcap_\alpha A_\alpha$ is closed. if A_α is closed.

Ex : $\bigcap_{n=1}^{\infty} \overline{D_{\frac{1}{n}}(x)} = \{x\}$.

$\{ |y-x| \le \frac{1}{n} \}$

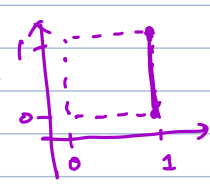
open: $\bigcup_{n=1}^{\infty} (n-1, n+1) \times (-\frac{1}{n}, \frac{1}{n})$

- Let $Z \subset \mathbb{C}$
- interior point : $z \in Z$ is an interior point, if $\exists U$ open nbhd of z. s.t. $U \subset Z$. [or. $\exists D_\varepsilon(z) \subset Z$]
- $\mathrm{int}(Z)$: set of all interior point.
- limit point : $z_0 \in \mathbb{C}$ is a limit point of Z, if there exist a seq of points in Z. $z_1, z_2, z_3, \cdots$, such that. $\lim_{i \to \infty} z_i = z$ (i.e. $\forall \varepsilon > 0$, $\exists N$ big enough, such. $\forall n > N$. $|z_n - z| < \varepsilon$.)
- $\overline{Z}$ = the set of limit points of. Z. (called the closure of Z)

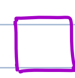
  " = smallest closed set containing Z."

  $= \bigcap_{\substack{A \text{ closed} \\ A \supset Z}} A$

Ex : $Z =$



- $\partial Z = \overline{Z} \setminus \mathrm{int}(Z)$.

$\partial Z =$  (all 4 edges).

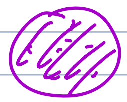
- **compact set** :

- bounded set : it can be covered by $D_R(0)$ for R large enough.

(less general) • $K \subset \mathbb{C}$ is a compact subset, if it is closed and bounded.

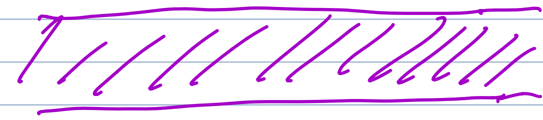
Ex:

unit closed disk.



$= \overline{D}$

$D = D_1(0)$



$\mathbb{R} \times [0,1]$.

closed but not bounded.

$\Rightarrow$ not compact.

Equivalently:

(1) • sequential compactness : K is seq. compact, if for any sequence in K, $z_1, z_2, \cdots$, there is a subsequence $z_{i_1}, z_{i_2}, z_{i_3}, \cdots$ ($i_1 < i_2 < i_3 \cdots$)

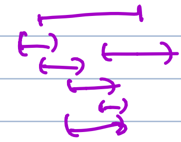
such that $\lim\limits_{k \to \infty} z_{i_k}$ exists and is in K.

(2) • (finite open cover.) : K is compact, if for any open covering of K, i.e $\{ U_\alpha \text{ open}, \ \alpha \in I \}$, $\underbrace{K \subset \bigcup_{\alpha \in I} U_\alpha}$

then, there exists a finite subset $I_0 \subset I$, such that.

$$K \subset \bigcup_{\alpha \in I_0} U_\alpha$$

Ex $K = [0, 1]$. $\subset \mathbb{R}$



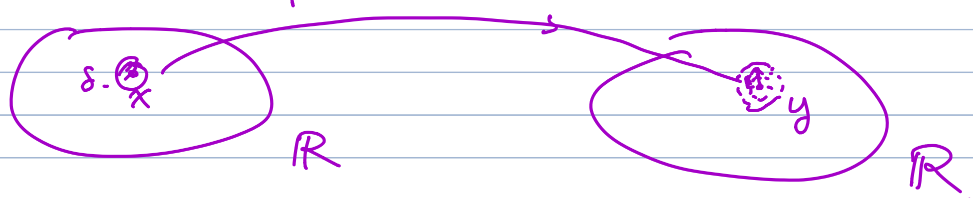
- Continuous Function : $f : \underset{\text{source}}{\mathbb{R}} \to \underset{\text{target}}{\mathbb{R}}$

can be generalized to
$\mathbb{R}^n \to \mathbb{R}^m$.
$\mathbb{C} \to \mathbb{C}$.
$\Omega \to \mathbb{R}$.
$\Omega \to \mathbb{C}$
....

Def: $f : \mathbb{R} \to \mathbb{R}$ be a function, it is continuous if $\forall x \in \mathbb{R}$. $\forall \varepsilon > 0$. $\exists \delta > 0$.

s.t. $\forall x' \in \mathbb{R}$ $|x' - x| < \delta$

$$\Rightarrow |f(x') - f(x)| < \varepsilon.$$



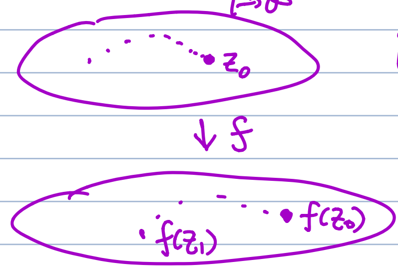
Equivalent definition for f being continuous: (1) for all open set $V \subset \mathbb{R}$ (target), $f^{-1}(V)$ is open.

"preimage of open is open" | only need open set

(2) "f preserves limit".

say $z_1, \cdots, z_n, \cdots, \in \mathbb{R}$ (source), s.t. $\lim_{i\to\infty} z_i = z_0$.

then. $f(z_0) = \lim_{i\to\infty} f(z_i)$.



$\mathbb{R}$ source

$\mathbb{R}$ target

Property: · If f is a continuous function, K is compact (in the source). then $f(K)$ is compact.

· C^0 (continuous), C^1, C^2, $\cdots$, C^k, $\cdots$.

C^∞ (smooth function) $\qquad C^\omega$: real analytic.

· Holomorphic Function:

· let $f : \Omega \to \mathbb{C}$, ($\Omega \subset \mathbb{C}$ open)

we say f is hol'c at $z_0 \in \Omega$, if the following limit exist.

$$\frac{f(z_0+h) - f(z_0)}{h} \quad \text{as } h \to 0, \quad h \in \mathbb{C}.$$

(if it exist, call it $f'(z_0)$).

· f is hol'c in Ω if f is hol'c at every point in Ω.

Ex: (1) $f : \mathbb{C} \to \mathbb{C}$. $f(z) = z^n$. (n integer. > 0).

$$f(z+h) - f(z) = (z+h)^n - z^n = \cancel{z^n} + n \cdot h \cdot z^{n-1} + O(h^2) - \cancel{z^n}.$$

$$= n \cdot h \cdot z^{n-1} + O(h^2)$$

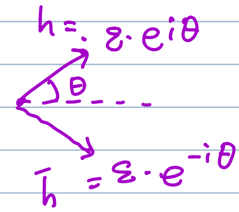
$$\lim_{h \to 0} \frac{n\, h \cdot z^{n-1} + O(h^2)}{h} = \lim_{h\to 0} n \cdot z^{n-1} + O(h) = n \cdot z^{n-1}.$$

$z = x + iy$

$\bar{z} = x - iy$

(2). $f : \mathbb{C} \to \mathbb{C}$. $\quad f(z) = \bar{z}$,

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{z+h} - \bar{z}}{h} = \frac{\bar{z} + \bar{h} - \bar{z}}{h}$$



$$= \frac{\bar{h}}{h}.$$

write $h = \varepsilon \cdot e^{i\theta}$, then $\bar{h} = \varepsilon \cdot e^{-i\theta}$

$$\frac{\bar{h}}{h} = \frac{\varepsilon \cdot e^{-i\theta}}{\varepsilon \cdot e^{i\theta}} = e^{-2i\theta}$$

$\Rightarrow$ $\lim(\cdots)$ of $h \to 0$ does not exist.

(3). $\frac{1}{z}$ is hol'c on $\mathbb{C} \setminus \{0\}$.

$\bar{z}^n$ is not hol'c.

$z \cdot \bar{z}$ is not hol'c.

- View $f : \mathbb{C} \to \mathbb{C}$ as $F : \mathbb{R}^2 \to \mathbb{R}^2$. $(x,y) \mapsto (u,v)$

  $\mathbb{C} \simeq \mathbb{R}^2$ $\qquad z = x + iy$. $\qquad z \leftrightarrow (x,y)$.

- Suppose F is C^1, differentiable. i.e.

  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exists and continuous.

[ex.: $f = \bar{z}$, $(x,y) \mapsto (x,-y)$]

$P \in \mathbb{R}^2$, $\quad H \in \mathbb{R}^2$.

$$\frac{F(P+H) - F(P)}{H} \quad \times$$

cannot (in general), divide a vector by a vector.

$\Uparrow$ F is real differentiable as $\mathbb{R}^2 \to \mathbb{R}^2$

- $F(P+H) = F(P) + T \cdot H + H \cdot \Phi(H)$ ($\because$)

T: 2×2 matrix; $\Phi(H) \to 0$ as $|H| \to 0$.

$$T = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = dF\Big|_P$$

- Now, let's use the fact f is hol'c.

$P = (x,y)$

$z = x+iy$.

$f(z) = u + iv$.

let $h = h_1 + i h_2$.

- h is real: $h = h_1$

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x+h_1, y) + iv(x+h_1, y) - (u(x,y) + i v(x,y))}{h_1}$$

$$= \frac{u(x+h_1, y) - u(x,y)}{h_1} + i \frac{v(x+h_1, y) - v(x,y)}{h_1}$$

$$\xrightarrow{h_1 \to 0} \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = f'(z).$$

- h is purely imaginary, $h = i \cdot h_2$.

$$\frac{u(x, y+h_2) - u(x,y)}{ih_2} + i \frac{v(x, y+h_2) - v(x,y)}{ih_2}$$

$$\xrightarrow{h_2 \to 0} \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = f'(z).$$

$\Rightarrow$ Cauchy Riemann

$$\left\{ \begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned} \right. \quad \Longleftarrow \quad f \text{ being hol'c at point } z.$$

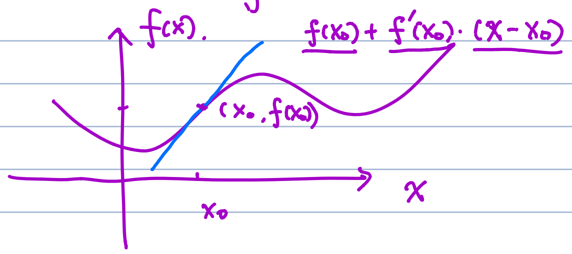
evaluated at point z.

$\boxed{\text{ex: } f(x,y) = \sqrt{|x| \cdot |y|} \text{ at point } 0.}$

u, v being differentiable + u, v satisfies CR. $\Longrightarrow$ f is hol'c.

Thm 2.4

• real 1-dim. $f: \mathbb{R} \to \mathbb{R}$



$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \underbrace{o(|x - x_0|)}$$

$\frac{\text{a term}}{|x-x_0|} \to 0$

as $x \to x_0$.

complex 1-dim: f is hol'c at z_0.

$\Downarrow$

$$f(z) = f(z_0) + f'(z_0)\cdot(z - z_0) + o(|z - z_0|).$$