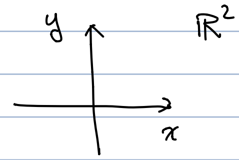
Complex Analysis in one variable.



$z = x + iy$.

- complex analytic functions : $f: \mathbb{C} \to \mathbb{C}$.

w/o "analytic", f can be viewed as $\mathbb{R}^2 \to \mathbb{R}^2$.

being analytic means, we can take derivative of f in the complex sense :

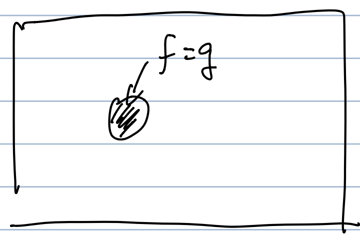
$$\frac{df}{dz}(z_0) := \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

h can be complex.
$\Rightarrow$ strong constraint.

analytic means $\frac{df}{dz}$ exist for all z_0.

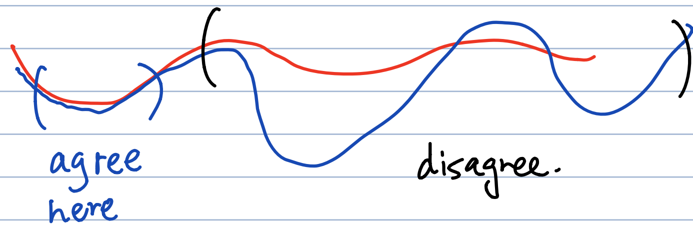
Consequence of Analyticity : "unique extension property"

① "local determines global" : if f and g are two analytic functions, and $f = g$ on a small open disk in $\mathbb{C}$

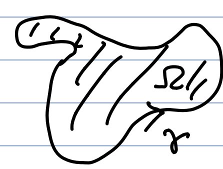


then $f = g$ everywhere.

[not the case for real function (non-analytic function).



② Regularity : even though we only require the 1st "complex" derivative to exist, it implies f is infinitely differentiable. (i.e. a smooth function.)



③ Cauchy (contour) integral : given a curve $\gamma \subset \mathbb{C}$. (satisfy certain mild property. e.g. γ encloses a region Ω and f is analytic on Ω), then $\int_\gamma f \cdot dz = 0$

Particular objects to study: (nice ~~an~~ analytic function).

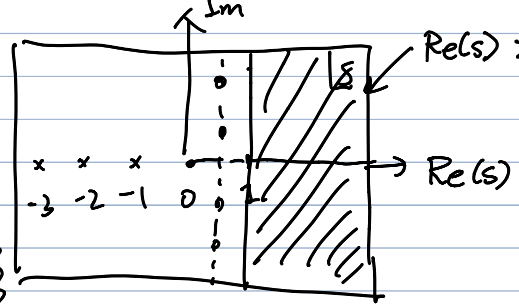
- Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

  - can $s = 2$? $\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ (converge !).
  - converge for $s > 1$, $s \in \mathbb{R}$.
  - converge for $s = x + iy$, $x, y \in \mathbb{R}$. $x > 1$.

$$2^{x+iy} = \left(e^{(\ln 2)}\right)^{(x+iy)} = e^{\ln 2 \cdot x} \cdot \underbrace{e^{\ln 2 \cdot iy}}$$

  - $\zeta(s)$ can be extended to be a analytic function everywhere on $\mathbb{C} \setminus \{\text{some points}\}$

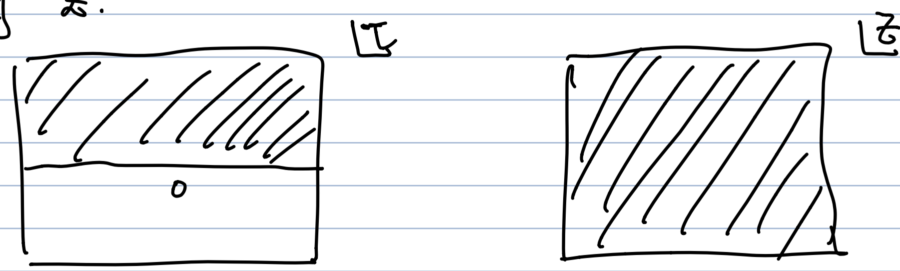


  - very useful in number theory. Riemann Hypothesis says: $\zeta(s)$ has all its zeros on the line $\mathrm{Re}(s) = \frac{1}{2}$.

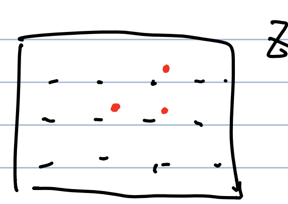
- Theta function:

$$\Theta(\tau \mid z) = \sum_{n=-\infty}^{+\infty} e^{2\pi i \cdot \left(\underbrace{\frac{1}{2} n^2 \cdot \tau}_{\text{dominate.}} + n \cdot z\right)}, \quad \text{say } \tau = i, \ i \cdot \tau = -1$$

  - the sum will converge for $\underline{\mathrm{Im}(\tau)} > 0$. (upper half plane). for arbitrary z.



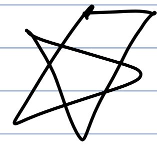
  - analytic in τ and z.
    - doubly periodic in z. (lattice depends on τ).



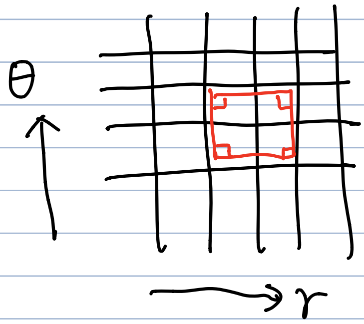
• "modular" in τ.

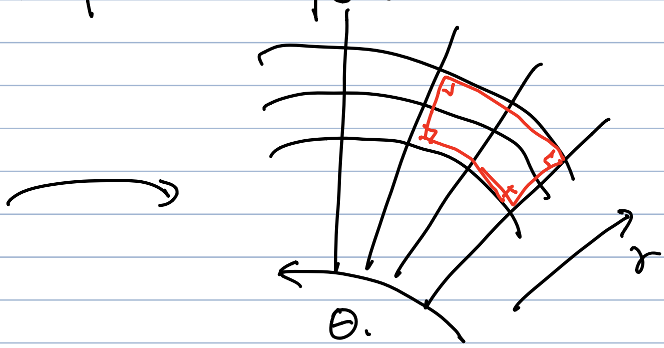
• Other topics :

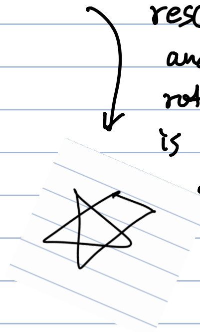
• Conformal Mapping :



conformal = " keep the shape ".

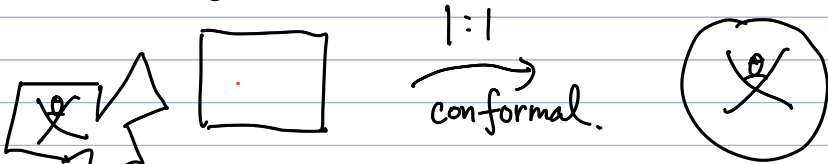




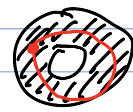


rescale and rotate is an example of conformal

Amazing Result "<u>Uniformization Thm</u>".



unit disk.

any <u>simply connected</u> (exclude things like )

region on $\mathbb{C}$.

$\mathbb{R}^n$: exist notion of conformal transformation, but we don't that much flexibility to bend any shape into one fixed shape.

<u>Basic</u> : • complex number. $+, -, \cdot, /, |\cdot|, \arg$.

(think of making $\mathbb{R}^3$ a number system ?)

• converge., limit., geometric pictures.

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notation.

x, y, u, v. real numbers,

z, w complex numbers,

t, s. real number.

• $\mathbb{C} \cong \mathbb{R}^2$. $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$.

• $z = x + iy$. $x, y \in \mathbb{R}$. $\boxed{i^2 = -1.}$

x : Real part , y: Imaginary part.

• addition , subtraction : $(x+iy) + (u+iv)$

$= (x+u) + i(y+v)$.

- $(x+iy)(u+iv) = xu + iy\cdot u + x\cdot(iv) + (iy)(iv)$
$= (xu - yv) + i(yu + xv).$

division: assume $x+iy \neq 0$ (i.e. either $x \neq 0$ or $y \neq 0$).

then. $\dfrac{u+iv}{x+iy} = (\ \cdot\ ) + i(\ \cdot\ )$?

$$= \frac{(u+iv)(x-iy)}{(x+iy)(x-iy)} = \frac{(ux+vy)+i(vx-uy)}{x^2+y^2}$$

$$= \frac{ux+vy}{x^2+y^2} + i\,\frac{vx-uy}{x^2+y^2}$$

("/")

- This makes $\mathbb{C}$ a division ring. ($+, -, \cdot$) $0, 1$, commutativity.
$z\cdot w = w\cdot z.$

- other examples of division ring. is

quaternion: $\mathbb{H} \simeq \mathbb{R}^4$. $q = x + iy + j\cdot u + k\cdot v.$

$i^2 = j^2 = k^2 = -1$ $x, y, u, v \in \mathbb{R}.$

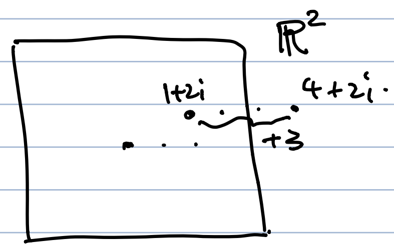
$ij = -ji.$ $i\cdot j = k.$ $i . j . k$

$ik = -ki$ $jk = i$

$\cdots$ $k\cdot i = j$

- Visualize. $\mathbb{C}$:

① using a plane



⊗ addition ⇝ translation.

$(1+2i) + (3) =$

multiplication ⇝. rotation + scaling

polar expression: $z = re^{i\theta}$ $r \geq 0$

$z' = r'\cdot e^{i\theta'}$ $\theta \in \mathbb{R} \mod 2\pi\mathbb{Z}.$

$\theta \sim \theta + 2\pi\cdot n.$

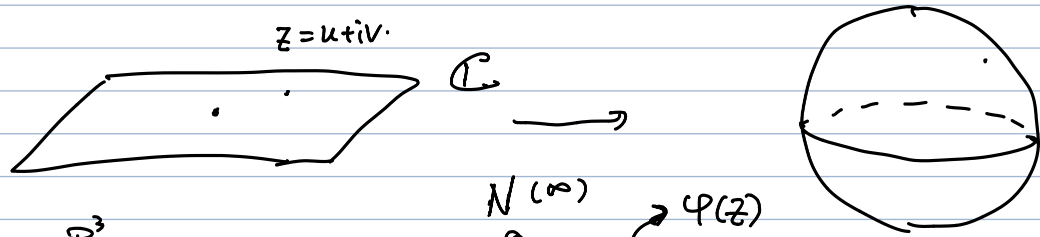
$z\cdot z' = (r\cdot r')\cdot e^{i(\theta+\theta')}$

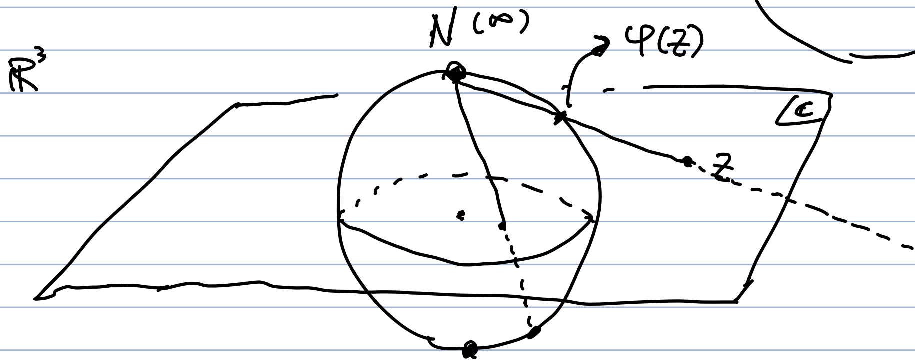
· r : absolute value (or, "modulus") of z

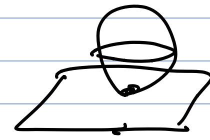
θ : argument (or, "phase") of z.

(see Ahlfors §1.2).

② Stereographic Projection : $x_1^2 + x_2^2 + x_3^2 = 1$.







$z \longmapsto \varphi(z)$

$\mathbb{C} \ni z$, $\varphi(z) \in S^2$

Benefit for this presentation:

- to include ∞ in the picture.
- to not discriminate $\underline{\underline{\infty}}$.

Read: : Stein : 1.1. + ($\underline{\underline{1.2}}$ ↙ convergence.)

Ahlfors : (Ch 1 ~~2~~)

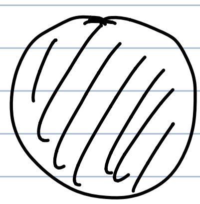
compactification:

$\mathbb{C} \longrightarrow S^2$ (1-pt compactification)

non-compact: $(0,1)$, $\mathbb{R}$, $\mathbb{R}^n$.

compact: $[0,1]$,  S^1

add one S^1



like a closed disk., interior = original $\mathbb{C}$.

· $f(z) = \dfrac{z+3}{z-1}$ ⇝ wants to live on S^2 ∵ $f(\infty) = 1$.

· e^{z^2} ...