Math 185 Second Midterm Nov 10, 2020

Name: ______________________________

- You have 80 minutes to complete the exam, 9:40-11:00am.
- Please write your name and page number on every page that you submit. The submission deadline is at 11:10am.
- This is a open-book exam, you can use your textbook and notes. You cannot use internet or calculator.
- You may only use the results covered in class so far, including results in the lecture note and results in Stein up to Chapter 2.
- If you have question during the exam, you may contact me in zoom,
- Please write neatly. Answers which are illegible for the reader cannot be given credit.

Good Luck!

Question	Points	Score
1	20	
2	40	
3	20	
4	20	
Total	100	

1. (20 points, 4 points each)

   (1) What is the definition of a residue?

   (2) What is the argument principle? What is winding number?

   (3) If f is a rational function on $\widehat{\mathbb{C}}$, is it always true that the total number of zero equals the total number of poles (considering multiplicity)?

   (4) What is open mapping theorem?

   (5) What is the definition of a simply connected domain? Is the following open set simply connected?

$$\Omega = \mathbb{D}\backslash\{x + iy \mid y = 0, x \in (-1, -0.5] \cup [0.5, 1)\}$$

2. (40 points, 10 points each) Show your steps in these calculations, otherwise no credits will be given.

   (1) Compute the residue of $f(z) = (z+1)^2/(z-1)^3$ at $z = 1$. (Hint: change variable, and let $u = z - 1$, compute the residue at $u = 0$).

   (2) Compute the number of zeros of $f(z) = z^3 - 4z + 1$ **inside and outside** the unit circle respectively.

   (3) Compute the integral $\int_{|z|=2} \frac{1}{z^5-1} dz$. (Hint: you do not need to compute the residues at the 5 roots of unity)

   (4) Let $f(z) = \frac{z^2(z+10)^3}{(z-10)^4}$, compute $\frac{1}{2\pi i}\int_{|z|=1} \frac{f'(z)}{f(z)} dz$.

3. (20 points) "Generalized residue theorem." If we compare the function with simple pole $\frac{1}{z-a}$ as the 'electric force' generated by a point charge at a, then the following is the analog of of charge distributed along a line segment $[a, b]$ with charge density $g(x)$ (assume to be smooth and bounded on the interval)

$$f(z) = \int_a^b \frac{g(x)}{z - x} dx$$

Let γ be a simple closed curve containing $[a, b]$ in the interior. If you like, you may choose $[a, b] = [-1/2, 1/2]$ and γ to be the unit circle.

(1) (15 pts) prove that

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \int_a^b g(x)dx$$

where the right hand side can be interpreted as the total charge inside γ.

The following part (2) and (3) might be a bit tricky, and one should consider them after finishing the rest of the exam.

(2) (3 pts) after the initial success, you decide to use a more powerful singularity than simple pole, and define the function

$$F(z) = \int_a^b \frac{g(x)}{(z-x)^2} dx$$

Compute the same integral $\frac{1}{2\pi i} \int_\gamma F(z) dz$.

(3) (2 pts) If you only know $f(z)$, can you recover $g(x)$? If so, describe your method.

4. (20 points) If a is a complex number with $|a| < 1$, then

(1) 5pt. show that $f(z) = \log(1 - az)$ is a single valued holomorphic function for $z \in \mathbb{D}$, with $f(0) = 0$.

(2) 15 pts. prove that

$$\int_0^{2\pi} \log|1 - ae^{i\theta}| d\theta = 0.$$