Today:

① review line integral from last time.

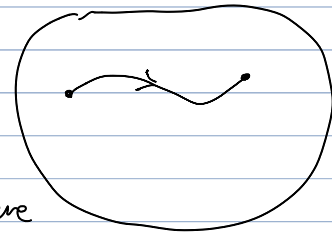
② $\int_\gamma F'(z)\,dz = F(\text{end pt}) - F(\text{start pt})$ if F hol'c and F' continuous.

③ Goursat thm: $\int_\nabla f\,dz = 0$, if f hol'c.

Review: $\int_{\gamma} f\,dz$ γ: integration path, curve.

- parametrized curve:
  - $\gamma : [a,b]_t \longrightarrow \mathbb{C}$, $t \longmapsto z(t) = x(t) + iy(t)$ $\Bigg\{$ • $(C^1\text{-})$ smooth continuous differentiable • or piecewise smooth.
  - such that $\gamma'(t) \neq 0$ for all $t \in [a,b]$.

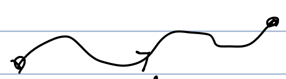
- equivalent parametrized curve:
  - $\tilde{\gamma} : [c,d]_s \longrightarrow \mathbb{C}$, $s \longmapsto \tilde{z}(s)$.



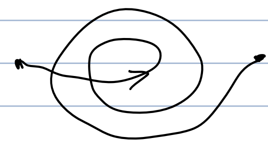
we say γ and $\tilde{\gamma}$ are equivalent, if there exist a map $\varphi : [c,d] \longrightarrow [a,b]$, $s \longmapsto t(s)$, that is continously diff, $t'(s) > 0$. and,

$$\gamma(t(s)) = \tilde{\gamma}(s).$$

- A curve is an equivalence class of parametrized curves.
- piecewise smooth: $\gamma : [a,b] \to \mathbb{C}$. $a = a_0 < a_1 < \cdots < a_n = b$, such that $\gamma|_{[a_i, a_{i+1}]}$ is smooth.
- a curve is simple, if there is no self-intersection. (except possibly at the end points)

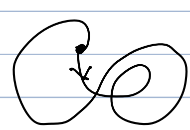


simple



not simple.

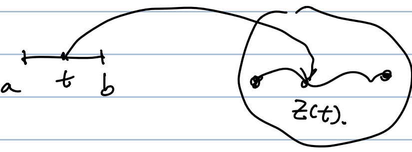
- a curve is closed, if its start point = finish point. $\gamma(a) = \gamma(b)$, $\gamma : [a,b] \to \mathbb{C}$.



- Integration of a function: Ω: region in $\mathbb{C}$
open connected subset.

$f: \Omega \to \mathbb{C}$ continuous.

$\gamma: [a,b] \to \Omega$. a curve
$t \mapsto z(t)$



$$\int_\gamma f\, dz := \int_a^b f(z(t)) \cdot z'(t) \cdot dt.$$

- To check that the definition doesn't depend on the parametrization, we take another equivalent parametrization: $\tilde{\gamma}: [c,d]_s \to \Omega$, $s \mapsto \tilde{z}(s)$.

$$\int_\gamma f\, dz = \int_a^b f(z(t))\, z'(t) \cdot dt$$

$$= \int_{s:c}^d f(z(t(s))) \cdot z'(t(s)) \cdot \frac{dt(s)}{ds} \cdot ds. \quad \text{plug in } t \text{ by } t(s).$$

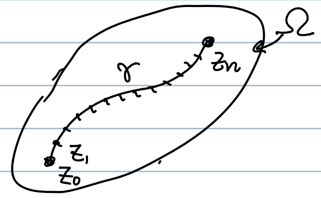
$z(t(s)) = \tilde{z}(s)$.

$$\frac{dz(t(s))}{ds} = \frac{dz}{dt} \cdot \frac{dt}{ds} \qquad = \int_c^d f(\tilde{z}(s)) \cdot \frac{d\tilde{z}}{ds} \cdot ds.$$

$\gamma: [a,b] \to \mathbb{R}$
$\downarrow \searrow \mathbb{C}$
$\mathbb{R}^n$.

$$= \int_{\tilde{\gamma}} f \cdot dz \quad (\text{param by } \tilde{\gamma}).$$

- Remarks: ① line. integral



$$\int_\gamma f\, dz = \lim_{n\to\infty} \cdot \sum f\left(\frac{z_i + z_{i-1}}{2}\right) \cdot (z_i - z_{i-1})$$

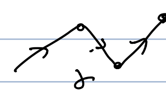
↑ # of segments. ↑ increment is a complex number.

② $f(z) = u(z) + i v(z)$, $dz = dx + i \cdot dy$.

$$\int_\gamma (u(z) + i v(z))(dx + i\, dy)$$

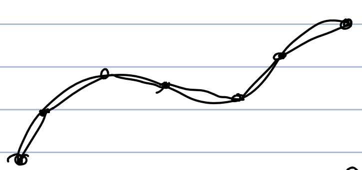
$$= \int_\gamma u \cdot dx - v\, dy + i \int_\gamma u \cdot dy + v\, dx$$

③. Generalized to: • γ being piecewise smooth



• "rectifiable arc": $\gamma: [a,b] \to \mathbb{C}$. ~~smth~~ continuous.

such that.
$$\left(\sup_{\substack{n.\\ a < t_1 < t_2 < \cdots < t_n = b}} \sum_{i=1}^n |\gamma(t_i) - \gamma(t_{i-1})| \right) < \infty.$$



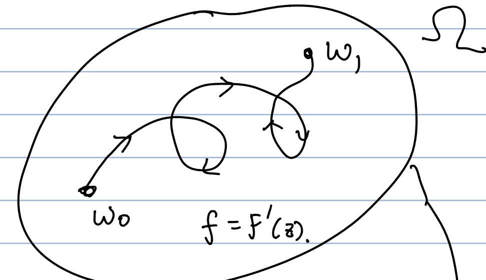
$\Leftrightarrow$ $\gamma(t) = x(t) + i y(t)$.
$x(t)$, $y(t)$ satisfies

$$\left(\sup \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| \right) < \infty \quad \text{and for } y.$$

$x(t)$. is of Bounded Variation.

② · **Thm** : If f is continuous function on Ω, and γ is a (piecewise C'-smooth) curve in Ω, start at w_0, end at w_1. and if f has a primitive, i.e. $\exists\ F:\Omega\to\mathbb{C}$. hol'c, and $F'(z)=f$, then.

$$\int_\gamma f\,dz = F(w_1)-F(w_0).$$



Rmk : • Definition of holomorphicity.

F is hol'c in $\Omega \iff \forall z\in\Omega$, F is hol'c at z, i.e. $F'(z)=\lim\limits_{h\to 0}\dfrac{F(z+h)-F(z)}{h}$ exist.

For Now:

$F':\Omega\to\mathbb{C}$ is just a function.

so far we don't even know if F' is continuous.

$\gamma(t)=z(t)$.

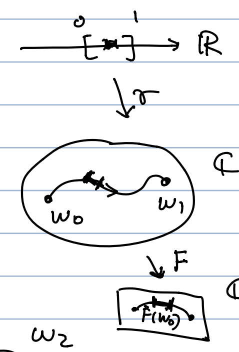
Pf : • Suppose γ is smooth., with parametrization.

$\gamma:[0,1]\to\Omega$.

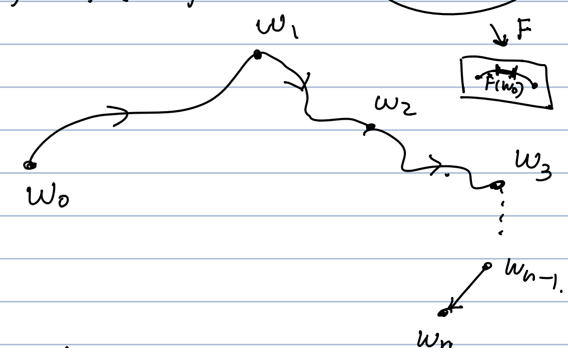
$\dfrac{dF}{dz}=f$

$$\int_0^1 f(\gamma(t))\cdot\gamma'(t)\cdot dt = \int_0^1 \overbrace{\left.\frac{dF}{dz}\right|_{\gamma(t)}\cdot\frac{d\gamma}{dt}\cdot dt}^{dF}$$

$$= F(\gamma(1)) - F(\gamma(0)) = F(w_1)-F(w_0).$$



• If γ is piecewise smooth,



$$\int_\gamma f\cdot dz = \sum_{i=0}^{n-1} F(w_{i+1})-F(w_i)$$

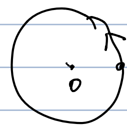
$$= F(w_n)-F(w_0) \quad \#$$

• **Cor** : If γ is a closed curve in Ω, $f:\Omega\to\mathbb{C}$ continuous and has a primitive F. then.

$$\int_\gamma f\,dz = 0.$$

Non-Example to the Cor :

- γ : unit circle in $\mathbb{C}$, oriented Counter Clockwise (CCW)



$$[0, 2\pi] \longrightarrow \mathbb{C}$$
$$\theta \longmapsto e^{i\theta}.$$

- $f(z) = \frac{1}{z}$

$$\int_\gamma f(z) \cdot dz = \int_0^{2\pi} f(e^{i\theta}) \cdot \frac{d e^{i\theta}}{d\theta} \cdot d\theta = \int_0^{2\pi} \frac{1}{e^{i\theta}} \cdot i \cdot e^{i\theta} \cdot d\theta$$

$(z = e^{i\theta})$
$$= \int_0^{2\pi} i \cdot d\theta = i \cdot 2\pi.$$

$$\oint_{|z|=1} \frac{1}{z} dz = 2\pi i. \qquad \left(\oint_{|z|=r} \frac{1}{z} dz = 2\pi i \right).$$

- The integral is not zero, because for whatever region Ω containing γ, there is no hol'c function F on Ω, s.t. $F' = f$.

(why not let $F(z) = \ln z$?).
not a single valued function along the circle.

$z = e^{\rho + i\theta}$ ambiguity in here by $2\pi n$.

$\ln z = \rho + i\theta$ not single valued.

- Summarize :
  - Holomorphic function in Ω := complex derivative exists for all $z \in \Omega$.
  - Analytic function in Ω. := $\forall z_0 \in \Omega$, there is a Taylor series expansion for f in a small disk around z_0.

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots$$
in a $D_\varepsilon(z_0)$.

- Analytic Function $\Rightarrow$ Holomorphic Function

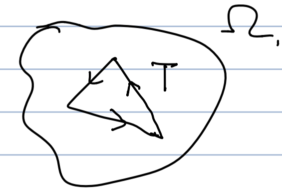
(derivative of power series can be taken termwise.)

- To go in the other direction, one need to use Cauchy integral.

$\int_\gamma f \, dz$ $\uparrow$ hol'c.

<u>Gaursat Thm</u>:

- If $f: \Omega \to \mathbb{C}$ is a hol'c function,
- if T is a (boundary of) triangle in Ω.

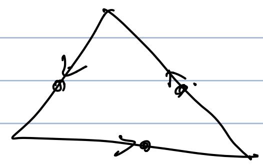


Then $$\int_T f \, dz = 0.$$

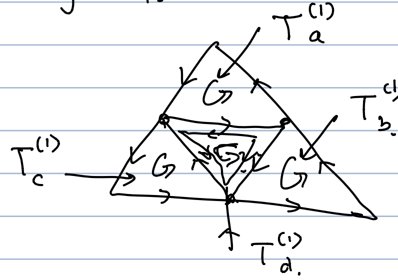
bold face

if T is triangle, use $\boldsymbol{T}$ for the solid triangle. (the closed set. bounded by T).

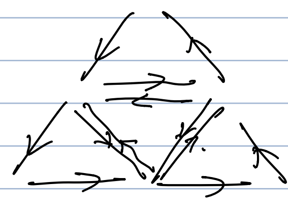
<u>Pf</u>: define subdivision of T.



$T = T^{(0)}$.



$$\int_T f \, dz = \int_{T_a^{(1)}} f \, dz + \int_{T_b^{(1)}} \cdots + \int_{T_c^{(1)}} (\cdots) + \int_{T_d^{(1)}} (\cdots)$$



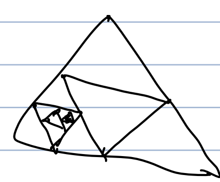
interior edge contribution cancel out.

let $T^{(1)}$ be the triangle, where $\left| \int_{T_i^{(1)}} f \, dz \right|$ is maximum.

$$\left| \int_{T^{(0)}} f \, dz \right| \leq \left| \int_{T_a^{(1)}} f \, dz \right| + \cdots + \left| \int_{T_d^{(1)}} f \, dz \right|.$$

$$\leq \underline{4} \cdot \left| \int_{T^{(1)}} f \, dz \right|.$$

similarly, define $T^{(2)}, T^{(3)}, \cdots$.



- let d_i = diameter of $(T^{(i)})$.

  P_i = perimeter of $T^{(i)}$.

$d_i = \frac{1}{2} \cdot d_{i-1}$, $p_i = \frac{1}{2} \cdot p_{i-1}$.

- There exist a unique point z_0, inside all of $T^{(i)}$.
- near z_0, we have

$$f(z) = \underbrace{f(z_0)}_{\text{const}} + \underbrace{f'(z_0)(z-z_0)}_{\text{linear term}} + \underbrace{(z-z_0)\cdot\psi(z)}_{?}$$

($\because$ f is hol'c at z_0.)

- $$\left|\int_{T^{(n)}} f(z)\, dz\right| = \left|\int_{T^{(n)}} (z-z_0)\cdot\psi(z)\cdot dz\right|$$

$$\le d_n \cdot p_n \cdot \varepsilon_n. \qquad \varepsilon_n := \sup_{z\in T_n} \psi(z)$$

$\therefore$ - - - - - - (to be continued).