Today:

1) Overview of Ch2

2) Goursat Theorem : $\int_{\triangle} f\, dz = 0$ ("entry point").

3) Cauchy Thm in a disk

1) Overview of Ch 2 (and review of Ch 1).

So far, what do we know about hol'c function?

① definition : complex derivative $f'(z)$ exist. ($f'(z)$ continuous?)

② convergent power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is hol'c.

(do all holomorphic functions admit such expansion?)

In Ch 2, we answer these questions in affirmative., and much more.

and a powerful is using line integral.

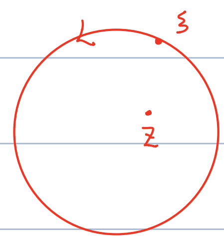
Main Results:

① $\oint_\gamma f(z)\, dz = 0$ if $f(z)$ is hol'c in $\mathbb{D}$ (unit open disk),

γ is a closed curve in $\mathbb{D}$.

(Thursday)

② Cauchy integral formula.



if f is hol'c on $\overline{\mathbb{D}}$, $C = \partial\mathbb{D}$, then.

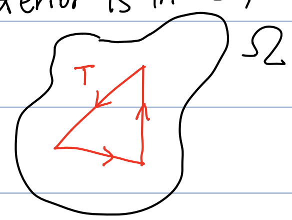
then, $$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z}\, d\xi.$$

" knowledge of the function at the boundary determines its behavior in the interior."

To begin, we start from a baby example of (1)., where γ is a triangle. , that is the Goursat thm.

Thm (Goursat): Let f be a hol'c function on a region Ω, and T be a (hollow) triangle in Ω, whose interior is in Ω, then

$$\int_T f(z)\,dz = 0$$



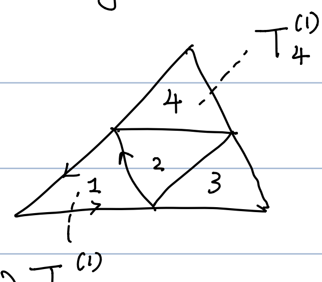
Idea: subdivide the triangle into smaller and smaller ones recursively, and prove by contradiction.

Pf: Let $T^{(0)} = T$ be the initial triangle.

$d^{(0)}$ = diamter of $T^{(0)} = \max\limits_{x,y\in T} |x-y|$

$p^{(0)}$ = perimeter of $T^{(0)}$ = sum of the lengths of 3 sides.

Divide $T^{(0)}$ into 4 parts.



$$T^{(0)} = T_1^{(1)} + T_2^{(1)} + T_3^{(1)} + T_4^{(1)}.$$

(if we write both sides as sums of oriented segments)

$$\therefore \left|\int_{T^{(0)}} f(z)\,dz\right| = \left|\sum_{i=1}^{4} \int_{T_i^{(1)}} f(z)\,dz\right| \le \sum_{i=1}^{4} \left|\int_{T_i^{(1)}} f(z)\,dz\right|.$$

$$\le 4 \cdot \max_{i=1,\cdots,4} \left|\int_{T_i^{(1)}} f(z)\,dz\right|$$

We let $T^{(1)}$ be a $T_i^{(1)}$ that achieves the above maximum.

Repeat the above process, one get a sequence of triangles $T^{(2)}, T^{(3)}, \dots$. Then

$$d^{(n)} = \text{diameter of } T^{(n)} = \frac{1}{2}\cdot d^{(n-1)}$$

$$= \frac{1}{4} d^{(n-2)} = \frac{1}{2^n}\cdot d^{(0)}$$

$$p^{(n)} = \text{perimeter of } T^{(n)} = \frac{1}{2^n}\cdot p^{(0)}.$$

② Let $\mathbb{T}^{(n)}$ (bold face) be the closed triangle, such that $T^{(n)}$

is the boundary. Then

$$\mathbb{T}^{(0)} \supset \mathbb{T}^{(1)} \supset \mathbb{T}^{(2)} \supset \cdots \quad \text{whose diameter} \to 0.$$

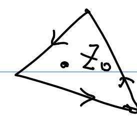
is a nested sequence of compact set. Then there exists a unique point z_0, $\in \mathbb{T}^{(n)}$ for all n.

(Pf: pick any $w_n \in \mathbb{T}^{(n)}$. then $\{w_n\}$ is a Cauchy sequence. hence converges to w. By closedness of $\mathbb{T}^{(n)}$, the limit point $w \in \mathbb{T}^{(n)}$ $\forall n$. That's existence. Now suppose there are 2 limit pts, w, w', then $\operatorname{diam}(\mathbb{T}^{(n)}) \geq |w-w'|$ for all n. contradicts with diam $\to 0$.)

③ We use the holomorphicity of $f(z)$ at z_0, to write

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \psi(z)\cdot(z-z_0).$$

where $|\psi(z)| \to 0$ as $z \to z_0$.


$T^{(n)}$

In particular, $\varepsilon_n := \max\limits_{z\in T^{(n)}} |\psi(z)| \to 0$ as $n\to\infty$

$$\int_{T^{(n)}} f\, dz = \int_{T^{(n)}} f(z_0) + f'(z_0)\cdot(z-z_0) + \psi(z)(z-z_0)\, dz.$$

$$= \int_{T^{(n)}} \frac{d}{dz}\left\{ f(z_0)\cdot(z-z_0) + f'(z_0)\cdot\frac{(z-z_0)^2}{2}\right\}\cdot dz + \int_{T^{(n)}} \psi(z)(z-z_0)\, dz$$

0 $\because \oint F'(z)\, dz = 0.$

$$\left|\int_{T^{(n)}} \psi(z)\cdot(z-z_0)\cdot dz\right| \leq \int_{T^{(n)}} |\psi(z)|\cdot|z-z_0|\cdot|dz|$$

$$\leq \max_{z\in T^{(n)}} |\psi(z)| \cdot \max_{z\in T^{(n)}} |z-z_0| \cdot \int_{T^{(n)}} |dz|$$

$$= \varepsilon_n \cdot d^{(n)} \cdot p^{(n)} = \frac{1}{4^n}\cdot \varepsilon_n \cdot$$

④ Put everything together. :

$$\left| \int_{T^{(0)}} f(z)\, dz \right| \leqslant 4^n \cdot \left| \int_{T^{(n)}} f(z)\, dz \right| \leqslant 4^n \cdot \varepsilon_n \cdot \frac{1}{4^n} \cdot p^{(0)} \cdot d^{(0)}$$

$$= \varepsilon_n \cdot p^{(0)} \cdot d^{(0)}.$$

since this is true for any n, we may let $n \to \infty$, to get

$$\left| \int_{T^{(0)}} f(z)\, dz \right| \leqslant 0. \Rightarrow \int_{T^{(0)}} f(z) \cdot dz = 0.$$

#.

With Goursat thm in hand, we immediately get

Corollary: Let Ω be open, $f: \Omega \to \mathbb{C}$ hol'c.
P be a polygon. (can be triangulated) whose interior is in Ω, then $\int_P f(z)\, dz = 0$

Notation: if γ is a closed curve, we sometimes write $\oint_\gamma f(z)\, dz$ for $\int_\gamma f(z)\, dz$.

(

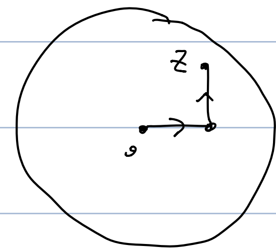
uni- open disk.

For simplicity and concreteness, let's consider hol'c on $\mathbb{D}$

Thm: Let f be a hol'c function on $\mathbb{D}$, then there exists a primitive F for f.

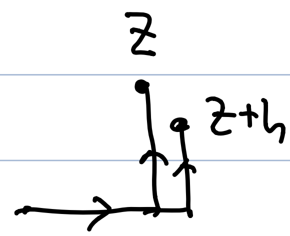
Pf: We construct a function $F: \mathbb{D} \to \mathbb{C}$, then we show that F is hol'c with $F' = f$.

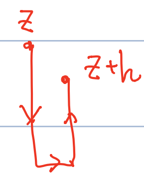
For any $z \in \mathbb{D}$, we define a curve γ_z from 0 to z as following (first move horizontally, then move vertically). And we define



$$F(z) = \int_{\gamma_z} f(w)\, dw.$$

- To check $F(z)$ is hol'c., we consider the small variations $F(z+h) - F(z)$.





$$= \int_{z \to z+h} f(w)\, dw$$

$$= \int f(w)\, dw + \int_{z \to z+h} f(w)\, dw$$

$$= 0 + \int_{z \to z+h} f(w)\, dw.$$

$\uparrow \because$ ▨ is contain in $\mathbb{D}$ and f is hol'c in $\mathbb{D}$

$\therefore$ by Goursat thm. & cor. vanishes.

$$= \int_{z \to z+h} f(z) + f'(z)(w-z) + \psi(w)(w-z)\, dw.$$

little "o" notation. represent a term $R(h)$ such that $\frac{R(h)}{h} \to 0$ as $h \to 0$.

$$= f(z) \cdot h + o(h)$$

$$= F(z+h) - F(z).$$

Thus, $F'(z) = \lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z)$.

<u>Thm</u>: (Cauchy's thm for a disk):

Let f be a hol'c function in $\mathbb{D}$.

γ a (piecewise smooth) closed curve in $\mathbb{D}$, then

$$\oint_\gamma f(z)\, dz = 0$$

<u>Pf</u>: $\because$ $f(z)$ has a primitive F.

$\therefore$ $\oint_\gamma F'(z)\, dz = 0$.