Today:

1) Overview of Ch 2

2) Goursat Theorem : $\int_{\triangle} f\,dz = 0$ ("entry point").

3) Cauchy Thm in a disk

1) Overview of Ch 2 (and review of Ch 1).

So far, what do we know about hol'c function?

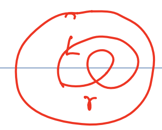
① definition : complex derivative $f'(z)$ exist. ($f'(z)$ continuous?)

② convergent power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is hol'c.

(do all holomorphic functions admit such expansion?)

In Ch 2, we answer these questions in affirmative., and much more.

and a powerful is using line integral.



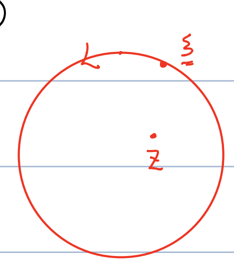
unit open disk

Main Results:

① $\oint_{\gamma} f(z)\,dz = 0$ if $f(z)$ is hol'c in $\mathbb{D}$, γ is a closed curve in $\mathbb{D}$.

(Thursday)

② Cauchy integral formula.



if f is hol'c on $\overline{\mathbb{D}}$, $C = \partial\mathbb{D}$, then.

then,
$$\boxed{f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{\xi - z}\,d\xi.}$$

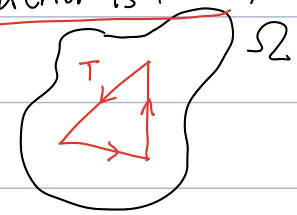
" knowledge of the function at the boundary determines its behavior in the interior."

To begin, we start from a baby example of (1)., where γ is a triangle. , that is the Goursat thm.

(Jordan thm: simple closed curve on $\mathbb{R}^2$, cut the plane into 2 parts: interior. exterior).

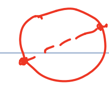
Thm (Goursat): Let f be a hol'c function on a region Ω, and T be a (hollow) triangle in Ω, whose interior is in Ω, then

$$\int_T f(z)\, dz = 0$$



Idea: subdivide the triangle into smaller and smaller ones recursively, and prove by contradiction.

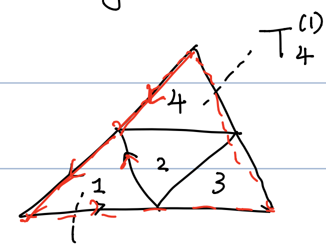
Pf: Let $T^{(0)} = T$ be the initial triangle.

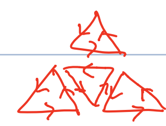


$d^{(0)}$ = diamter of $T^{(0)} = \max_{x,y\in T^{(0)}} |x-y|$

$p^{(0)}$ = perimeter of $T^{(0)}$ = sum of the lengths of 3 sides.

Divide $T^{(0)}$ into 4 parts.





$$T^{(0)} = T_1^{(1)} + T_2^{(1)} + T_3^{(1)} + T_4^{(1)}$$

(if we write both sides as sums of oriented segments) $T_2^{(1)}$

$$\therefore \left|\int_{T^{(0)}} f(z)\, dz\right| = \left|\sum_{i=1}^4 \int_{T_i^{(1)}} f(z)\, dz\right| \le \sum_{i=1}^4 \left|\int_{T_i^{(1)}} f(z)\, dz\right|.$$

$$\le 4\cdot \max_{i=1,\cdots,4} \left|\int_{T_i^{(1)}} f(z)\, dz\right|$$

We let $T^{(1)}$ be a $T_i^{(1)}$ that achieves the above maximum.

Repeat the above process, one get a sequence of triangles $T^{(2)}, T^{(3)}, \cdots$. Then

$$d^{(n)} = \text{diameter of } T^{(n)} = \frac{1}{2}\cdot d^{(n-1)}$$

$$= \frac{1}{4} d^{(n-2)} = \frac{1}{2^n}\cdot d^{(0)}$$

$$p^{(n)} = \text{perimeter of } T^{(n)} = \frac{1}{2^n}\cdot p^{(0)}.$$

② Let $\mathbb{T}^{(n)}$ (bold face) be the closed triangle, such that $T^{(n)}$

is the boundary. Then

$$\mathbb{T}^{(0)} \supset \mathbb{T}^{(1)} \supset \mathbb{T}^{(2)} \supset \cdots$$ whose diameter $\to 0$.

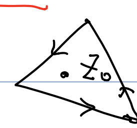
is a nested sequence of compact set. Then there exists a unique point z_0, $\in \mathbb{T}^{(n)}$ for all n.

(Pf: pick any $w_n \in \mathbb{T}^{(n)}$. then $\{w_n\}$ is a Cauchy sequence. hence converges to w. By closedness of $\mathbb{T}^{(n)}$, the limit point $w \in \mathbb{T}^{(n)}$ $\forall n$. That's existence. Now suppose there are 2 limit pts, w, w', then $\text{diam}(\mathbb{T}^{(n)}) \geq |w-w'|$ for all n. contradicts with diam $\to 0$.)

③ We use the holomorphicity of $f(z)$ at z_0, to write

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \psi(z)\cdot(z-z_0). \iff \frac{f(z)-f(z_0)}{z-z_0} \to f'(z_0)$$

where $|\psi(z)| \to 0$ as $z \to z_0$.



In particular, $\varepsilon_n := \max_{z \in T^{(n)}} |\psi(z)| \to 0$ as $n \to \infty$

they have primitives.

$$\int_{T^{(n)}} f\, dz = \int_{T^{(n)}} \left(f(z_0) + f'(z_0)\cdot(z-z_0) \right) + \psi(z)(z-z_0)\, dz.$$

$$= \int_{T^{(n)}} \frac{d}{dz}\left\{ f(z_0)\cdot\cancel{(z-z_0)} + f'(z_0)\cdot\frac{(z-z_0)^2}{2} \right\}\cdot dz + \int_{T^{(n)}} \psi(z)(z-z_0)\, dz$$

0 $\because \oint F'(z)\, dz = 0$.

$\gamma: [0,1] \to \mathbb{C}$

$\max|f| \int_0^1 |\gamma'(t)|\, dt \geq \left| \int_0^1 f(\gamma(t))\cdot\gamma'(t)\cdot dt \right| = \left| \int_\gamma f\, dz \right| \leq \left(\max_\gamma |f|\right)\cdot(\text{length } \gamma)$

$$\left| \int_{T^{(n)}} \psi(z)\cdot(z-z_0)\cdot dz \right| \leq \int_{T^{(n)}} |\psi(z)|\cdot|z-z_0|\cdot|dz|$$

ds

$$\leq \max_{z \in T^{(n)}} |\psi(z)| \cdot \max_{z \in T^{(n)}} |z-z_0| \cdot \int_{T^{(n)}} |dz|$$

$$\leq \varepsilon_n \cdot d^{(n)} \cdot p^{(n)} = \frac{1}{4^n}\cdot \varepsilon_n \cdot d^{(0)}\cdot p^{(0)}$$

④ Put everything together :

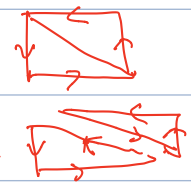
$$\left| \int_{T^{(0)}} f(z)\,dz \right| \leq 4^n \cdot \left| \int_{T^{(n)}} f(z)\,dz \right| \leq 4^n \cdot \underline{\varepsilon_n \cdot \frac{1}{4^n} \cdot p^{(0)} \cdot d^{(0)}}$$

$$= \underline{\varepsilon_n} \cdot p^{(0)} \cdot d^{(0)}.$$

since this is true for any n, we may let $n \to \infty$, to get

$$\left| \int_{T^{(0)}} f(z)\,dz \right| \underline{\leq} 0. \Rightarrow \int_{T^{(0)}} f(z)\,dz = 0.$$

#.



With Goursat thm in hand, we immediately get

Corollary: Let Ω be open, $f: \Omega \to \mathbb{C}$ hol'c.

P be a polygon. (can be triangulated) whose interior is in Ω, then $\int_P f(z)\,dz = 0$

Notation: if γ is a closed curve, we sometimes write $\oint_\gamma f(z)\,dz$ for $\int_\gamma f(z)\,dz$.

(

uni- open disk.

For simplicity and concreteness, let's consider hol'c on $\mathbb{D}$

$f = u + iv$

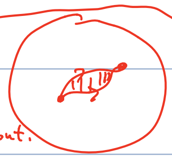
$\int (u+iv)(dx+idy)$

Thm: Let f be a hol'c function on $\mathbb{D}$, then there exists a primitive F for f.

p, q are C^1

$pdx + qdy$

if $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$ and is cont. then

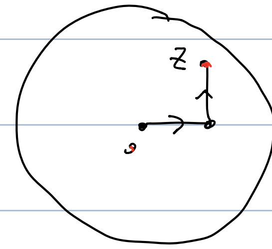


$\int_\gamma pdx + qdy$ only dep

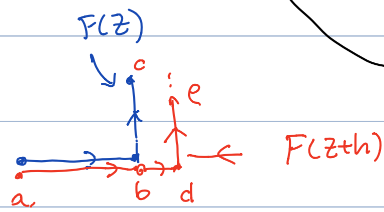
Pf: We construct a function $F: \mathbb{D} \to \mathbb{C}$, then on the endpt.

we show that F is hol'c with $F' = f$.

For any $z \in \mathbb{D}$, we define a curve γ_z from 0 to z as following (first move horizontally, then move vertically). And we define



$$F(z) := \int_{\gamma_z} f(w)\, dw.$$

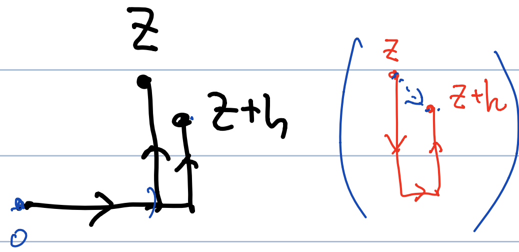


$F(z) = \int_{a \to b} + \int_{b \uparrow c}$

$F(z+h) = \int_{a \to b \to c} + \int_{b \to c} + \int_{c \uparrow d}$

• To check $F(z)$ is hol'c., we consider the small variations

$F(z+h) - F(z)$.



$$= \int_{z \downarrow \to \uparrow z+h} f(w)\, dw$$

$$= \int_{\square} f(w)\, dw + \int_{z \to z+h} f(w)\, dw$$

$$= 0 + \int_{z \to z+h} f(w)\, dw.$$

$\left| \int_\gamma f\, dz \right| \le \max|f| \cdot |\text{length } \gamma|$

$\uparrow$ $\because$ ▨ is contain in $\mathbb{D}$ and f is hol'c in $\mathbb{D}$

$\therefore$ by Goursat thm. & cor. vanishes.

$\int_{z \to z+h} f(z) \cdot dw = f(z) \cdot \int_{z \to z+h} 1\, dw$

$= f(z) \cdot (z+h-z)$

$= f(z) \cdot h.$

$f'(z) \cdot \frac{h^2}{2}$

$$= \int_{z \to z+h} \left(f(z) \right) + \left(f'(z)(w-z) + \psi(w)(w-z) \right) dw.$$

little "o" notation.

represent a term $R(h)$ such that $\frac{R(h)}{h} \to 0$ as $h \to 0$.

$$\boxed{= \left(f(z) \cdot h \right) + o(h)}$$

$= F(z+h) - F(z)$.

Thus, $F'(z) = \lim\limits_{h \to 0} \dfrac{F(z+h) - F(z)}{h} = f(z)$.

\# "Existence of primitive for hol'c function on a disk".

Thm: (Cauchy's thm for a disk):

Let f be a hol'c function in $\mathbb{D}$.

γ a (piecewise smooth) closed curve in $\mathbb{D}$, then

$$\Rightarrow \oint_\gamma f(z)\, dz = 0$$

Pf: $\because$ $f(z)$ has a primitive F. i.e., $F' = f$

$\therefore$ $\oint_\gamma F'(z)\, dz = 0$.

$\quad\quad \parallel$

$F(\gamma(b)) - F(\gamma(a))$ $\quad$ $\gamma : [a,b] \to \mathbb{D}$.

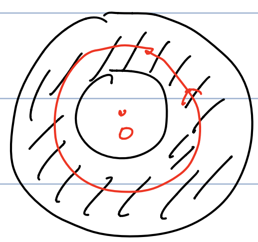
(Morera thm) $\quad$ $\gamma(a) = \gamma(b)$.

Rmk: Converse Thm: if f is continuous in $\mathbb{D}$.

if $\underline{\underline{\forall}}\ \gamma$ closed curve in $\mathbb{D}$, $\int_\gamma f\, dz = 0$

then f is hol'c.

$A = \{ \frac{1}{2} < |z| < 2 \}$

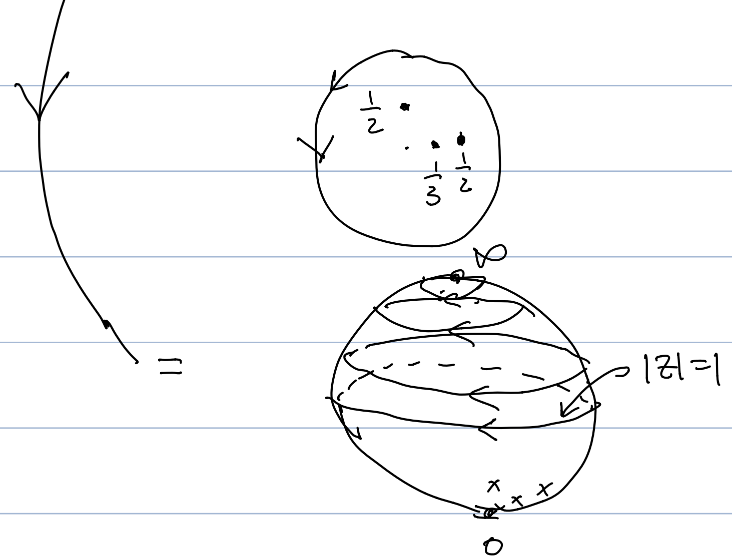
$A =$ 

$f : A \to \mathbb{C}$ $\quad$ $f(z) = \frac{1}{z}$.

$$\oint_{|z|=1} f(z) \cdot dz = 2\pi i = \oint \frac{1}{z}\, dz$$

Ex:

$$\oint_{|z|=1} \underbrace{\left(\frac{1}{z - \frac{1}{2}} + \frac{1}{z - \frac{1}{3}} + \frac{1}{z - \frac{i}{2}} \right)}_{f(z).} \cdot dz$$



change variable,

$w = \frac{1}{z}$.

so $z=\infty \iff w=0$.

$d\left(\frac{1}{w}\right) = \frac{-1}{w^2}\cdot dw$

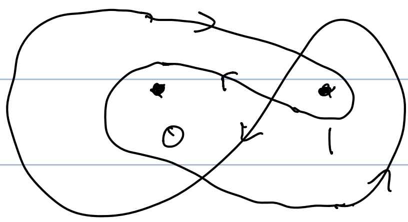
$f\left(\frac{1}{w}\right)\left(\frac{-1}{w^2}\right) = g(w)$.

$$= \oint f\left(\tfrac{1}{w}\right)\, d\tfrac{1}{w}$$

$$= \oint_{|w|=1} g(w)\, dw$$

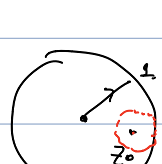
$$= 0$$

$g(w)$ is a hol'c function on $\{|w|\le 1\}$



$z^{\alpha}\cdot(z-1)^{\beta}$, $\quad \alpha,\beta\in\mathbb{C}$

#18:



need to show $f(z) \overset{?}{=} \sum_{n=0}^{\infty} c_n (z-z_0)^n$ for $z\in D_r(z_0)$

$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}(\cdots) = \sum_{j=0}^{\infty}\sum_{n=j}^{\infty}(\cdots)$ $\quad z = z_0 + w$

$w = z - z_0$

• $f(z) := \sum_{n=0}^{\infty} a_n\cdot z^n = \sum_{n=0}^{\infty} a_n(z_0+w)^n =$

$$\begin{array}{c|l} & \to j \\ n=0 & a_0 \\ n=1 & a_1(z_0+w) \\ n=2 & a_2(z_0^2+2wz_0+w^2) \\ \vdots \downarrow n & \vdots \end{array}$$

Q: given $|z_0|<R$.

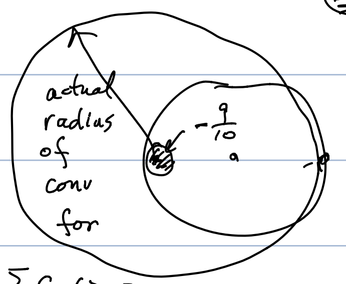
• can one find $\{c_n\}$ s.t. $f(z) = \sum_n c_n\cdot(z-z_0)^n$ for $|z-z_0|<r$?

c_n " $f^{(n)}(z_0)\frac{1}{n!}$

① $c_n = ?$ (ans: it will be an infinite sum)

② it converges, at least, for $|z-z_0|<r$

requires one to **bound** $|c_n|$



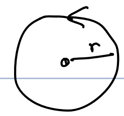
$\Sigma c_n (z-z_0)$.

$$f(z)=\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n \qquad R_0=1$$

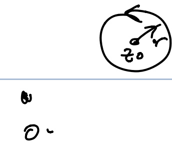
$$z_0=-\frac{1}{2}$$

$$f(z)=f(z_0)+c_1\cdot(z-z_0)+\cdots$$

#25 (a)



(b)



(c) hint: $\dfrac{1}{(z-a)(z-b)}=\dfrac{?}{z-a}+\dfrac{?}{z-b}$