Today: Stein §2.4 Cauchy integral formula

§2.3 Some integral examples.

Recall:

1. If $f: \Omega \to \mathbb{C}$ continuous, and $f = F'$ with F hol'c.

   then $\oint_\gamma f(z)\, dz = 0$ for γ closed curve in Ω.

   (need the existence of primitive)

2. Goursat Thm:

   - if f is hol'c in Ω, and γ is a $\underbrace{\text{triangle}}_{\text{in } \Omega}$, then

   $$\oint_\gamma f\, dz = 0$$

3. Existence of primitive for a <u>hol'c function</u> <u>in a disk</u>.

4. Cauchy theorem: (from 1. and 3.)

   If f is hol'c in the disk $\mathbb{D}$, then

   $$\oint_\gamma f\, dz = 0.$$ for all γ closed curve in Ω.

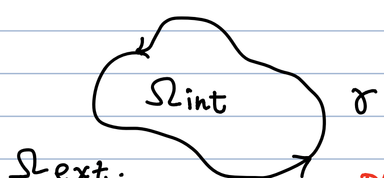
Today:

(I) Cauchy Thm for <u>simple closed curve.</u>

We will quote the result of Jordan's theorem:

- Let γ be a simple closed curve in $\mathbb{C}$, then there is an <u>interior</u> region Ω_{int} and an <u>exterior</u> region Ω_{ext}, such that

$\partial\Omega_{int} = \partial\Omega_{ext} = \gamma$, $\Omega_{int} \cap \Omega_{ext} = \phi$., $\mathbb{C} = \Omega_{int} \sqcup \Omega_{ext} \sqcup \gamma$.



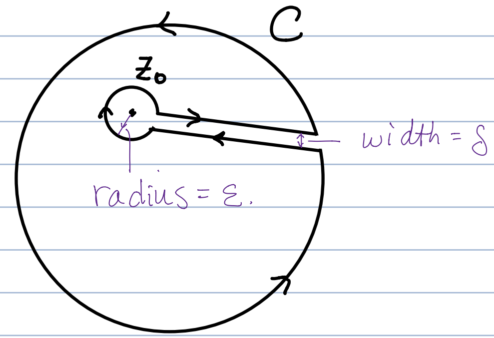
positively oriented, Ω_{int} is always on your left

"Toy contour": those simple closed curve such that Ω_{int} can be identified unambiguously.

- We say a toy contour is <u>oriented positively</u>, if you walk on the curve along the orientation direction, Ω_{int} is on your left.

- <u>Example</u>:

    - Key hole contour:

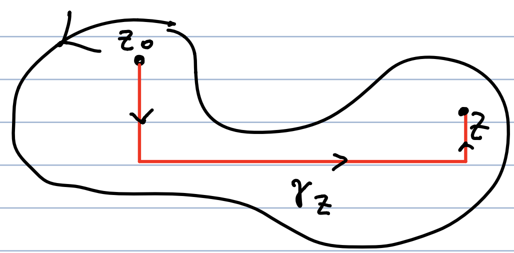


Let C be a circle, z_0 be a point enclosed in C, a keyhole contour $\Gamma_{\varepsilon,\delta}$ is a contour that defours to go around z_0 in an ε-radius circle.

(<u>terminology</u>: let $A \subset \mathbb{C}$ be a closed set, we say f is hol'c on A if there is an open nbhd $\mathcal{U}$ of A, s.t. f is hol'c on $\mathcal{U}$)

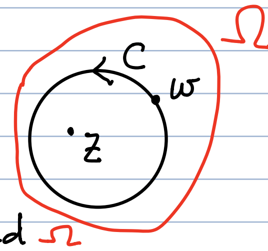
- <u>"Informal Thm"</u>: Let γ be a toy contour, Ω_{int} be the interior region, and f is hol'c on $\overline{\Omega_{int}}$, then there exists a holomorphic function F on $\overline{\Omega_{int}}$, such that $F' = f$ on $\overline{\Omega_{int}}$.

<u>Pf</u>: the same approach as γ is the unit circle, except when we define F by integration, more zigzaggy path is needed.



pick a base pt z_0

Cauchy Integral Formula (on a disk).



D: open unit disk

Thm: Let $C = \partial\mathbb{D}$ be the boundary of the unit disk, and f is hol'c on an open nbhd Ω of $\overline{\mathbb{D}}$, then for any $z \in \mathbb{D}$

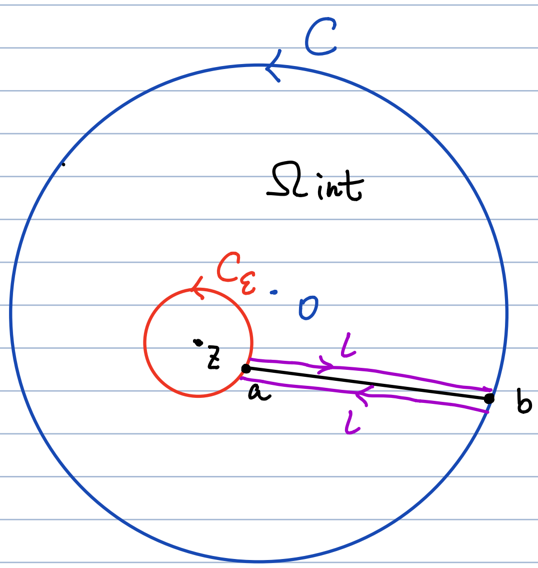
$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z}\, dw .$$

Strategy: we will deform C to a small circle of radius ε near z, and show that the integral is invariant. Then, we will let $\varepsilon \to 0$, and evaluate the integral

$$\frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \frac{f(w)}{w-z}\, dw .$$

Pf: ① We use the "key-hole" contour to show

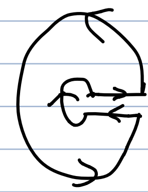
$$\frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i} \oint_{C_\varepsilon} \frac{f(w)}{w-z}\, dw$$



- C = circle of radius 1 centered at 0

C_ε = circle of radius ε centered at z

(notation differ from Stein).

$\Gamma_{\varepsilon,\delta}$ = 

let $\tilde{f}(w) = \dfrac{f(w)}{w-z}$, then $\tilde{f}$ is hol'c on $\overline{\Omega_{int}}$, hence.

$$\oint_{\Gamma_{\varepsilon,\delta}} \frac{f(w)}{w-z}\, dz = 0.$$

Let the corridor width $\delta \to 0$, and notice that the integral along the two segments on the corridor cancels out, we get the claim.

② Now, we prove that

$$\frac{1}{2\pi i} \oint_{C_\varepsilon} \frac{f(w)}{w-z}\, dw = f(z).$$

- $$\frac{f(w)}{w-z} = \frac{f(w)-f(z)}{w-z} + \frac{f(z)}{w-z}$$

$\because \lim_{w \to z} \frac{f(w)-f(z)}{w-z} = f'(z)$ $\therefore$ $\frac{f(w)-f(z)}{w-z}$ is bounded by M for $|w-z| \le \varepsilon_0$

Thus $\left| \oint_{C_\varepsilon} \frac{f(w)-f(z)}{w-z}\, dw \right| \leq M \cdot \text{length}(C_\varepsilon)$ for all $\varepsilon < \varepsilon_0$

since LHS is independent of ε, we may take limit $\varepsilon \to 0 \Rightarrow \text{LHS} \overset{\shortparallel}{0}$

- $$\frac{1}{2\pi i} \oint_{w \in C_\varepsilon} \frac{f(z)}{w-z}\, dw = f(z) \cdot \frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \frac{1}{w-z}\, dw = f(z).$$

#.

<u>Remark</u>: • we can replace unit circle C by any toy contour, theorem still holds.

• Key hole contour can be used to show that: " contour integral is invariant as we deform the contour γ within the region where the integrand is holomorphic."

• Can be generalized, even if $f(z)$ is not holomorphic on $\mathbb{D}$.

$$f(z) = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}} \frac{f(w)}{w-z}\, dw + \frac{1}{2\pi i} \iint_{\mathbb{D}} \frac{\partial f(w)}{\partial \bar{w}} \cdot \frac{1}{w-z}\, \underbrace{dw \wedge d\bar{w}}_{\overset{\shortparallel}{(-2i)\, dx \cdot dy}}$$

(see Griffiths-Harris, ch 0).

- §2.3 Sample Calculations.