Today: Stein §2.4 Cauchy integral formula

§2.3 Some integral examples.

Recall:

1. If $f: \Omega \to \mathbb{C}$ continuous, and $f = F'$ with F hol'c.

   then $\oint_\gamma f(z)\,dz = 0$ for γ closed curve in Ω.

   (need the existence of primitive)

2. Goursat Thm:

   - if f is hol'c in Ω, and γ is a triangle in Ω, then

   $$\oint_\gamma f\,dz = 0$$

3. Existence of primitive for a hol'c function in a disk.

4. Cauchy theorem: (from 1. and 3.)

   If f is hol'c in the disk $\mathbb{D}$, then

   $$\oint_\gamma f\,dz = 0.$$ for all γ closed curve in Ω.

Today:

(I) Cauchy Thm for simple closed curve.

We will quote the result of Jordan's theorem:

- Let γ be a simple closed curve in $\mathbb{C}$, then there is an interior region Ω_{int} and an exterior region Ω_{ext}, such that

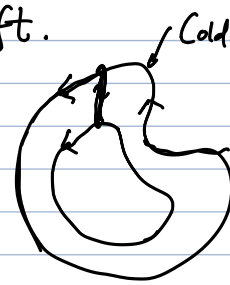
$$\partial\Omega_{int} = \partial\Omega_{ext} = \gamma, \quad \Omega_{int} \cap \Omega_{ext} = \phi., \quad \mathbb{C} = \Omega_{int} \sqcup \Omega_{ext} \sqcup \gamma.$$

simple // closed curve.

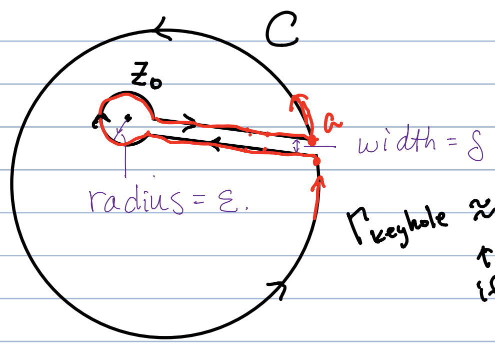
positively oriented, Ω_{int} is always on your left

"Toy contour": those simple closed curve such that Ω_{int} can be identified ~~unambiguously~~.

- We say a toy contour is <u>oriented positively</u>, if you walk on the curve along the orientation direction, Ω_{int} is on your left.



- <u>Example</u>:
  - Key hole contour:



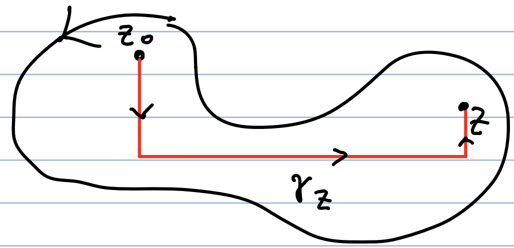
$\Gamma_{keyhole} \approx C_{old} - C_{new}$
↑
if $\delta \to 0$

Let C be a circle, z_0 be a point enclosed in C, a keyhole contour $\Gamma_{\varepsilon,\delta}$ is a contour that detours to go around z_0 in an ε-radius circle.

(<u>terminology</u>: let $A \subset \mathbb{C}$ be a closed set, we say f is hol'c on A if there is an open nbhd $\mathcal{U}$ of A, s.t. f is hol'c on $\mathcal{U}$)

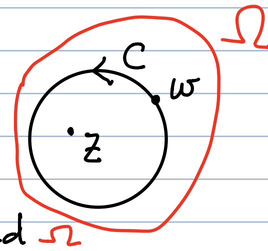
- <u>"Informal Thm"</u>: Let γ be a toy contour, Ω_{int} be the interior region, and f is hol'c on $\overline{\Omega_{int}}$, then there exists a holomorphic function F on $\overline{\Omega_{int}}$, such that $F' = f$ on $\overline{\Omega_{int}}$.

<u>Pf</u>: the same approach as γ is the unit circle, except when we define F by integration, more zigzaggy path is needed.



- pick a base pt z_0
- $F(z) = \int_{z_0}^{z} f(z) \cdot dz$

Cauchy Integral Formula (on a disk).



D: open unit disk

Thm: Let $C = \partial \mathbb{D}$ be the boundary of the unit disk, and f is hol'c on an open nbhd Ω of $\overline{\mathbb{D}}$, then for any $z \in \mathbb{D}$

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z}\, dw.$$

Strategy: we will deform C to a small circle of radius ε near z, and show that the integral is invariant. Then, we will let $\varepsilon \to 0$, and evaluate the integral

$$\frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \frac{f(w)}{w-z}\, dw.$$

Pf: ① We use the "key-hole" contour to show

$$\frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i} \oint_{C_\varepsilon} \frac{f(w)}{w-z}\, dw$$

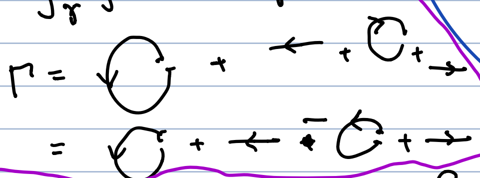
$\gamma : [0,1] \to \mathbb{C}$

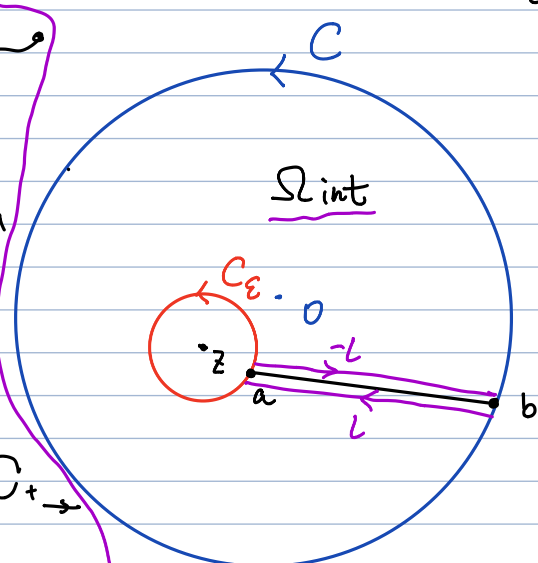
$\tilde{\gamma} : [0,1] \to \mathbb{C}$

$\tilde{\gamma}(t) = \gamma(1-t)$

(flip the orientation of γ).

$\cdot \int_\gamma f\, dz = -\int_{\tilde{\gamma}} f \cdot dz$

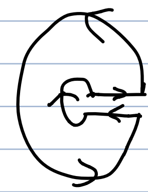




• C = circle of radius 1 centered at 0

C_ε = circle of radius ε centered at z

(notation differ from Stein).

$\Gamma_{\varepsilon,\delta}$ = 

Let $\tilde{f}(w) = \dfrac{f(\omega)}{\omega - z}$, then $\tilde{f}$ is hol'c on $\overline{\Omega_{int}}$, hence.

$$\oint_{\Gamma_{\varepsilon,\delta}} \frac{f(\omega)}{\omega - z}\, dz = 0.$$

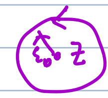
Let the corridor width $\delta \to 0$, and notice that the integral along the two segments on the corridor cancels out, we get the claim.

$$\oint_C \cdots + \oint_{\overline{C_\varepsilon}} \cdots = 0 \iff \oint_C \cdots = \oint_{C_\varepsilon} \cdots$$

(2) Now, we prove that

$$\frac{1}{2\pi i} \oint_{C_\varepsilon} \frac{f(w)}{w-z}\, dw = f(z).$$

• the value of the integral is ε- indep.



•
$$\frac{f(w)}{w-z} = \underbrace{\frac{f(w)-f(z)}{w-z}}_{(A)} + \underbrace{\frac{f(z)}{w-z}}_{(B)}$$

$\because \lim_{w \to z} \dfrac{f(w)-f(z)}{w-z} = f'(z)$ $\therefore$ $\dfrac{f(w)-f(z)}{w-z}$ is bounded by M for $0 < |w-z| \leq \varepsilon_0$

value is ε- indep.

Thus $\left| \oint_{C_\varepsilon} \dfrac{f(w)-f(z)}{w-z}\, dw \right| \leqslant M \cdot \text{length}(C_\varepsilon)$ for all $\varepsilon < \varepsilon_0$

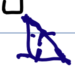
since LHS is independent of ε, we may take limit $\varepsilon \to 0 \Rightarrow$ LHS $= 0$

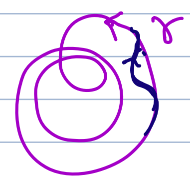
•
$$\frac{1}{2\pi i} \oint_{w \in C_\varepsilon} \frac{f(z)}{w-z}\, dw = f(z) \cdot \underbrace{\frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \frac{1}{w-z}\, dw}_{= \frac{1}{2\pi i} \oint_{|u|=\varepsilon} \frac{1}{u} du = 1} = f(z).$$ #

$u = w - z$

$$\oint_{|u|=\varepsilon} \frac{1}{u}\, du = \int_{\theta=0}^{2\pi} \frac{1}{\varepsilon \cdot e^{i\theta}}\, d(\varepsilon \cdot e^{i\theta}) = \int_{\theta=0}^{2\pi} \frac{1}{\cancel{\varepsilon \cdot e^{i\theta}}}\, \cancel{\varepsilon \cdot e^{i\theta}} \cdot i\, d\theta = i \cdot \int_0^{2\pi} d\theta = 2\pi i$$

<u>Remark</u>: • we can replace unit circle C by any toy contour, theorem still holds.

$\gamma - \gamma' =$ 



• Key hole contour can be used to show that:

" contour integral is invariant as we deform the contour γ within the region where the integrand is holomorphic. "

• Can be generalized, even if $f(z)$ is not holomorphic on $\mathbb{D}$.

$$f(z) = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}} \frac{f(w)}{w-z}\, dw + \frac{1}{2\pi i} \iint_{\mathbb{D}} \frac{\partial f(w)}{\partial \bar{w}} \cdot \frac{1}{w-z}\, \underbrace{dw \wedge d\bar{w}}_{= (-2i)\, dx \cdot dy}$$

(see Griffiths - Harris, ch 0).

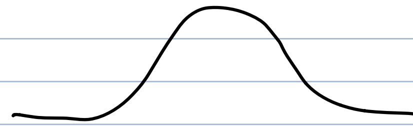
· §2.3 Sample Calculations.

· ξ is a parameter

· x is integral variable.

Ex: $$\int_{-\infty}^{+\infty} e^{-\pi x^2 - 2\pi i \cdot x \cdot \xi}\, dx \doteq e^{-\pi \cdot \xi^2}$$

· $e^{-\pi x^2}$, $e^{-x^2/2\sigma}$ Gaussian.

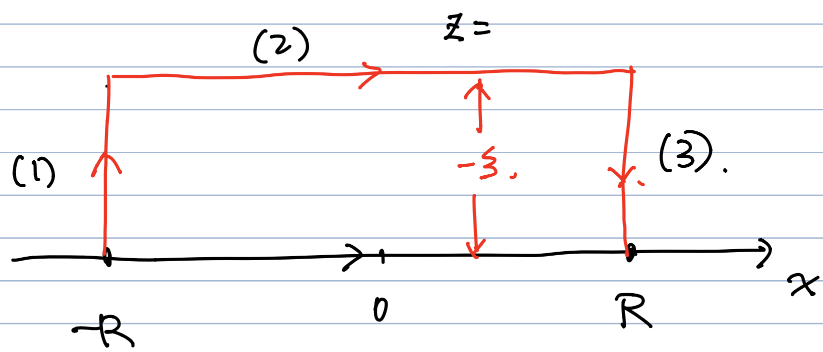


· $e^{-2\pi i \cdot x \cdot \xi}$ · Fourier transformation kernel.

· plane wave with vector ξ

$$= \lim_{R \to \infty} \int_{-R}^{R} \underline{e^{-\pi x^2 - 2\pi i \cdot x \xi}}\, dx.$$

↑ this function is hol'c in x.



$z = x + iy$

$y = -\xi$

$x \in [-R, R]$.

part(2). $$\int_{-R+i\xi}^{R+i\xi} e^{-\pi \cdot z^2 - 2\pi i \cdot z \cdot \xi}.\, dz$$

$$= \int_{-R}^{R} e^{\overset{+\pi\xi^2}{-\pi (x - i\xi)^2} - \overset{-2\pi\xi^2}{2\pi i (x - i\xi)\xi}}.\, dx$$

$$= \int_{-R}^{R} e^{-\pi x^2 - \pi \xi^2}\, dx$$

part (1) and (3).

$$(1) = \int_{-R}^{-R-i\xi} e^{-\pi z^2 - 2\pi i \cdot z \cdot \xi} \, dz$$

$z = x + iy$

x fixed, $=-R$

y goes from 0 to $-\xi$.

$$= \underbrace{e^{-\pi \cdot R^2 + C \cdot R + C'}}$$

this integral's absolute value is bounded by

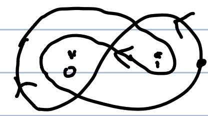
$$C \cdot e^{-\pi R^2 + c' R}.$$

Hence, $\lim_{R \to \infty} (1) + (3) = 0$.

$$\lim_{R \to \infty} (2) = \int_{-\infty}^{+\infty} e^{-\pi x^2 - \pi \xi^2} dx = e^{-\pi \xi^2} \cdot \underbrace{\int_{-\infty}^{+\infty} e^{-\pi x^2} \cdot dx}_{=1.}$$

$$= e^{-\pi \xi^2}.$$

look up "Gaussian integral"

- Pochhammer contour. 

$$\int_C z^{\alpha} (z-1)^{\beta} \, dz \qquad \alpha, \beta \in \mathbb{C}$$

- P12. Stein. $f(z) = u(x,y) + i v(x,y)$

$$\frac{\partial f}{\partial \bar{z}} = 0 \iff \frac{1}{2} \left(\partial_x f + i \partial_y f \right) = 0$$

$$\iff \frac{1}{2} \left(\partial_x (u + iv) + i \partial_y (u + iv) \right) = 0$$

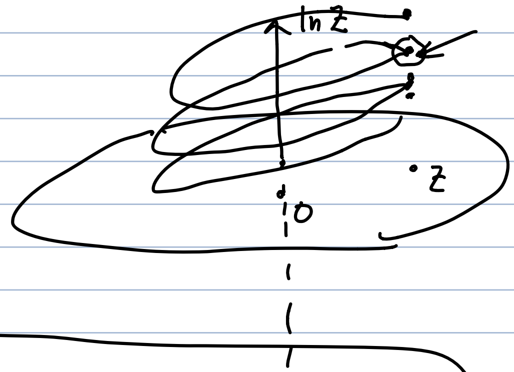
$$\iff (\partial_x u - \partial_y v) + i (\partial_x v + \partial_y u) = 0$$

$\Leftrightarrow$ CR.

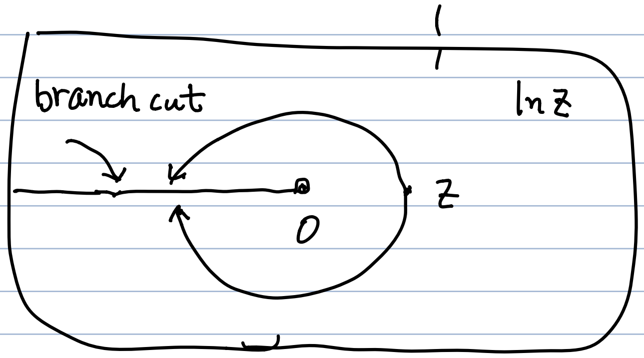
- $\ln z$ is a multivalued function on $\mathbb{C} \setminus \{0\}$.

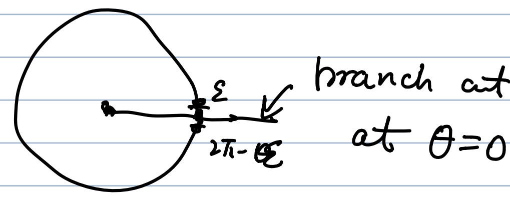
  if $z = r \cdot e^{i\theta}$ $r > 0$.

  then $\ln z = \ln r + i\theta$ up to addition of $i2\pi \cdot n$, $\underline{n \in \mathbb{Z}}$



$\mathbb{C}$





$$\int_C \frac{1}{z}\, dz = \int d \ln z$$

$$= \ln z \Big|_{1 \cdot e^{i\varepsilon}}^{1 \cdot e^{i(2\pi - \varepsilon)}}$$

$$= \left[\ln 1 + i(2\pi - \varepsilon)\right] - \left[\ln 1 + i(\varepsilon)\right]$$

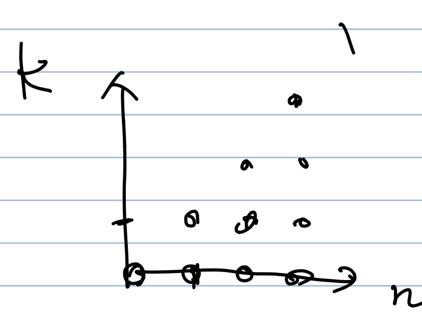
$$= 2\pi i - \underline{\underline{2i\varepsilon}}$$

$\sum a_n \cdot z^n$

$= \sum a_n (z_0 + u)^n$ $u = z - z_0$

$= \sum_n \underbrace{a_n} \sum_{k=0}^{n} \cdot \binom{n}{k} z_0^{n-k} \cdot u^k$

$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \underline{a_n \cdot \binom{n}{k} \cdot z_0^{n-k} \, u^k}$



need to show:

$\underline{\underline{(!)}}$ $\sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \left(a_n \binom{n}{k} z_0^{n-k}\right) u^k$

- this indeed converge
- but not enough to justify !.

- $$\oint_{|z|=1} \frac{1}{z-a}\, dz = \begin{cases} 2\pi i, & \text{if } \\ 0, & \text{if } \end{cases}$$