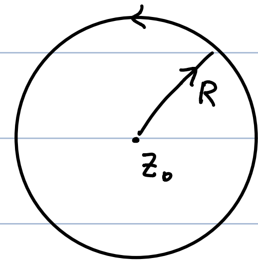
Cor 4.3. Cauchy Inequality (aka Cauchy estimate)

Let f be a hol'c function on an open nbhd of $\overline{\mathbb{D}}$. Then



$\mathbb{D} = \mathbb{D}_R(z_0)$

$C = \partial \mathbb{D}$.

$$\left| f^{(n)}(z_0) \right| \leq n! \, \frac{\|f\|_C}{R^n}$$

where $\|f\|_C = \max_{z \in C} |f(z)|$.

Pf: $$\left| f^{(n)}(z_0) \right| = \left| \oint_C \frac{f(w)}{(w-z)^{n+1}} \frac{n!}{2\pi i} \, dw \right| \leq \oint \frac{|f(w)|}{R^{n+1}} \cdot \frac{n!}{2\pi} |dw|$$

$$\leq \frac{\|f\|_C}{R^{n+1}} \frac{n!}{2\pi} \, 2\pi R = \frac{n!}{R^n} \cdot \|f\|_C. \quad \square.$$

Remark: • one can relax C to be any simple closed curve., then replace. $R = \min_{w \in C} d(z_0, w)$.

Thm 4.4. (Existence of Power Series expansion).

Let f be a holomorphic function on a region Ω.

$z_0 \in \Omega$, $\mathbb{D} = \mathbb{D}_R(z_0)$, $\overline{\mathbb{D}} \subset \Omega$, $C = \partial \mathbb{D}$. Then $\forall z \in \mathbb{D}$.

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z-z_0)^n$$

where $a_n = f^{(n)}(z_0) / n!$

$A \subset \mathbb{C}$ subset.

$\bar{A}$, A^{int}

$\partial A = \bar{A} \setminus A^{\text{int}}$

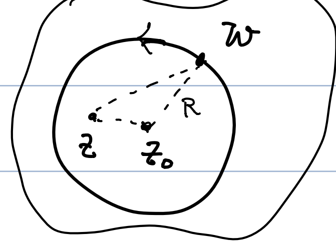
Pf: By Cauchy integral formula

$$f(z) = \oint_C \frac{f(w)}{w-z} \frac{dw}{2\pi i} = \oint_C \frac{f(w)}{w - z_0 - (z-z_0)} \frac{dw}{2\pi i}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$\oint \quad f(w) \quad \frac{1}{\quad} \quad \frac{dw}{\quad}$$

$$= \oint_C \frac{\quad}{w-z_0} \quad \frac{\quad}{1-\frac{z-z_0}{w-z_0}} \quad \frac{\quad}{2\pi i}$$



note that $\rho = \left|\frac{z-z_0}{w-z_0}\right| < 1$, thus,

$$f(z) = \oint_C \sum_{n=0}^{\infty} \frac{f(w)}{w-z_0} \left(\frac{z-z_0}{w-z_0}\right)^n \frac{dw}{2\pi i}$$

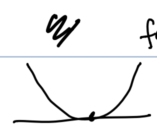
To show that the integral and summation can switch order, it suffices to show that

$$\oint \sum_n \left| \frac{f(w)}{w-z_0} \cdot \frac{(z-z_0)^n}{(w-z_0)^n} \right| \cdot \left|\frac{dw}{2\pi}\right| < \infty$$

$$\text{LHS} \le \frac{\|f\|_C}{R} \cdot \sum_n \rho^n \cdot \frac{2\pi R}{2\pi} = \|f\| \cdot \frac{1}{1-\rho} < \infty .$$

$$\therefore\ f(z) = \sum_{n=0}^{\infty} \oint_C \frac{f(w)}{w-z_0} \frac{(z-z_0)^n}{(w-z_0)^n} \frac{dw}{2\pi i}$$

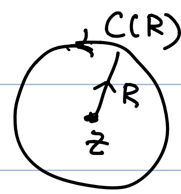
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$



$f(x) = e^{-\frac{1}{x^2}}$ $\quad x \in \mathbb{R}$. $\quad$ at $x=0$, $\quad f^{(n)}(x) = 0$. $\quad \forall n$.

near $x=0$. $\quad f(x) \neq \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

Cor: (Liouville Thm): if f is entire and bounded, then f is constant.



Pf: It suffices to show $f'(z) = 0$ for all z.

$$|f'(z)| \le \frac{\|f\|_{C(R)}}{R} \le \frac{1}{R}\left(\sup_{z\in\mathbb{R}} |f(z)|\right) \text{ for all } R.$$

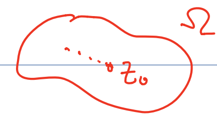
Hence $f'(z) = 0$.

$\sin(iy) = \frac{e^{-y} - e^{y}}{2i}$ $\qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$

Cor: Every degrees n polynomial has precisely n roots in $\mathbb{C}$. (Read Stein for a proof).

Let Ω be a connected open subset of $\mathbb{C}$.

<u>Thm 4.8</u> $f: \Omega \to \mathbb{C}$ hol'c. If there exist a convergent distinct point $z_1, z_2, \cdots$ in Ω, such that $z_0 = \lim z_n \in \Omega$. and $f(z_n) = 0$ $\forall n$. Then. $f(z) = 0$. $\forall z \in \Omega$.



<u>Pf</u>: Consider power series expansion near z_0, i.e. fix $\overline{D_r(z_0)} \subset \Omega$.

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z-z_0)^n \qquad \forall z \in D_r(z_0).$$

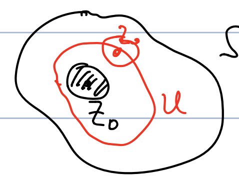
We claim that $a_n = 0$ for all n. Otherwise, let a_m be the first nonzero coeff. Then

$$f(z) = a_m (z-z_0)^m + a_{m+1} (z-z_0)^{m+1} + \cdots$$

$$= a_m (z-z_0)^m \Big(1 + \underbrace{(z-z_0) \cdot \sum_{n=0}^{\infty} \frac{a_{m+1+n}}{a_m} (z-z_0)^n}_{h(z).} \Big)$$

By continuity, $\forall \varepsilon > 0$, $\exists r \geq \delta > 0$. s.t. $\forall |z-z_0| < \delta$, $|(z-z_0) h(z)| < \varepsilon$.

We may choose ε small enough, s.t. $1 - \varepsilon > 0$. Hence, within $D_\delta(z_0)$ disk, $f(z)$ has no other zero than z_0, contradicting with $\lim_{n\to\infty} z_n = z_0$, z_n distinct, $f(z_n) = 0$.



We conclude that $f(z) = 0$ by continuity. Let

$$Z = \{ z \in \Omega \mid f(z) = 0 \}$$

$D_r(z_0) \subset Z$ open

and $\mathcal{U}$ be the <u>interior</u> of Z. $\mathcal{U}$ contains $D_r(z_0)$ above, hence is non-empty. <u>But $\mathcal{U}$ is also closed (in Ω)</u>, since if $z_0 \in \partial\mathcal{U}$, then we may chose $z_n \in \mathcal{U}$ distinct and converge to z_0. By the above argument, $f = 0$ in <u>an open disk</u> nbhd of z_0. Hence $z_0 \in \mathcal{U}$. Since $\mathcal{U}$ is open and closed in Ω, Ω is connected, hence $\Omega = \mathcal{U}$. ☒

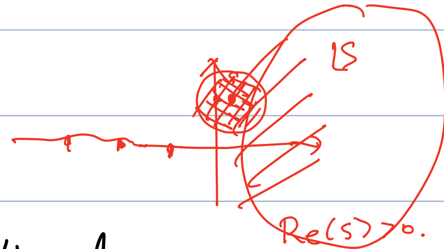
let $V = \Omega \setminus U$, $\because U$ closed in Ω. $\therefore V$ is open. $\Omega = U \sqcup V$ $\because \Omega$ is connected. $\therefore \Omega$ cannot be written as disjoint union of nonempty open. $\Rightarrow V = \emptyset$.

Cor: Let $f, g : \Omega \to \mathbb{C}$ be hol'c. If $f = g$ on an open subset $U \subset \Omega$, then $f = g$ on Ω.

Def: (Analytic continuation) Let Ω, Ω' be two regions, $\Omega \subset \Omega'$. $f : \Omega \to \mathbb{C}$, $F : \Omega' \to \mathbb{C}$ hol'c. And $F|_\Omega = f$. Then we say F is an analytic continuation of f.

Example

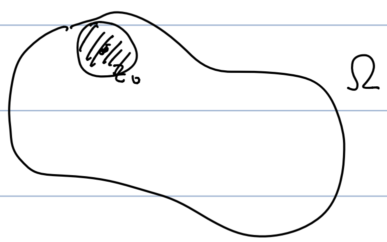
- $\Gamma(s) := \int_0^\infty e^{-t} \cdot t^{s-1}\, dt$ for $\mathrm{Re}(s) > 0$



but $\Gamma(s)$ can be analytically continued.

- there are functions that are hol'c on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, but cannot be extended any further.

Rmk:



$f : \Omega \to \mathbb{C}$ hol'c.

$\overline{D_r(z_0)} \subset \Omega$.

f has Taylor "power" series expansion around z_0.

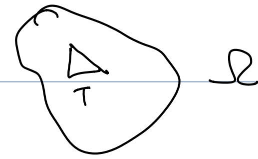
$$f(z) = \sum_{n=0}^\infty a_n (z - z_0)^n \quad \forall\ |z - z_0| < r.$$

Q: what is the radius of convergence for ? is it r?

A: radius of convergence is possibly bigger.

Thm (Morera): If $f : \Omega \to \mathbb{C}$ continuous, and if for all triangles T in Ω.

(*) $\int_T f \, dz = 0$



Then f is holomorphic in Ω.

Pf: the hypothesis (*) is the output of Goursat theorem.

thus, we can find primitive F for f, i.e. $F' = f$.

Now F is hol'c $\qquad F^{(n)}$ is hol'c. $\Rightarrow$ f is hol'c.

$\Downarrow \qquad\qquad\qquad \Uparrow$

F is analytic $\Rightarrow$ $F^{(n)}$ is analytic.

Sequence of hol'c functions.

Thm:

- $f_1, f_2, f_3, \cdots$ be a sequence of hol'c on Ω.

- assume that $\lim_{n\to\infty} f_n(z) = f(z)$ uniformly on every compact subset of Ω.

  - $\forall K \subset\subset \Omega$ compact. $\forall \varepsilon > 0$, $\exists N$, s.t. $\sup_{\substack{z \in K \\ n > N}} |f_n(z) - f(z)| < \varepsilon$.

Then: f is hol'c in Ω.

Pf: We are going to show that, for all T triangles in Ω,

$$\int_T f \, dz = 0.$$

$\because$ T is a compact set. $f_n \rightrightarrows f$ on T uniformly.

$\therefore$ $$\int_T f \, dz = \lim_{n\to\infty} \int_T f_n \, dz = 0.$$

$\therefore$ By Morera thm, f is hol'c.