Today: • Finish power series.

• begin integration along curves.

Power series:

• Definition: $\sum_{n=0}^{\infty} a_n \cdot z^n$.

radius of convergence R: if $|z| < R$, converge absolutely; if $|z| > R$, diverge.

How to determine R?

• $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. Hadamard formula.

• if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exist, then it equals $\frac{1}{R}$. (problem 17).

• Convergence of functions: $f_1, f_2, f_3, \cdots$ a seq of fun on a domain Ω.

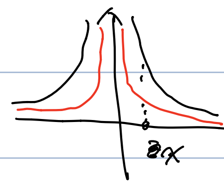
- different mode of convergence:

• pointwise convergence: $\forall z \in \Omega$, $f_1(z), f_2(z), f_3(z), \cdots \to$

• uniform convergence: $\forall \varepsilon > 0$, $\exists N$. (indep of z), s.t.

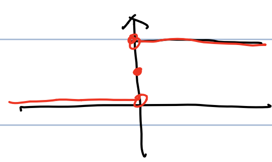
$\sup_{z\in\Omega} |f_i(z) - f(z)| < \varepsilon$. $\forall i > N$.

(not uniform convergence example: $f_n(x) = \frac{1}{n \cdot x^2}$



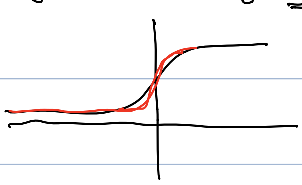
on $\Omega = (0, \infty)$. $f_n(x)$ converges pointwise to zero. but not uniformly).

(not uniform conv. ex:

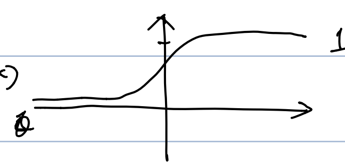


$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$

$f_n(x) = \text{erf}(n \cdot x)$.



erf(x)



$= \mathbb{P}(X < x)$ X: normal

$= \int_{-\infty}^{x} e^{-t^2/2} \frac{dt}{\sqrt{2\pi}}$.

under pointwise convergence, we cannot preserve continuity).

Thm (2.6): If $\sum_{n=0}^{\infty} a_n \cdot z^n$ converges, with radius of conv. $0 < R$, and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\{|z|<R\}$. Then $f(z)$ is holomorphic, and its derivative is given by (taking derivative termwise).

$$g(z) = \sum_{n=0}^{\infty} a_n \cdot n \cdot z^{n-1}$$ (could have started with $n=1$).

and this serie has radius of convergence R.

proof: (1) check $\sum_{n=0}^{\infty} a_n \cdot n \cdot z^{n-1}$ has the same radius of conv.

$$\limsup_{n\to\infty} |a_n \cdot n|^{\frac{1}{n}} = \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right) \cdot \left(\lim_{n\to\infty} n^{\frac{1}{n}}\right) = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

claim: $\lim_{n\to\infty} n^{\frac{1}{n}} = 1 \iff \lim_{n\to\infty} \log(n^{\frac{1}{n}}) = 0$

$\iff \lim_{n\to\infty} \frac{1}{n} \log n = 0.$ ✓.

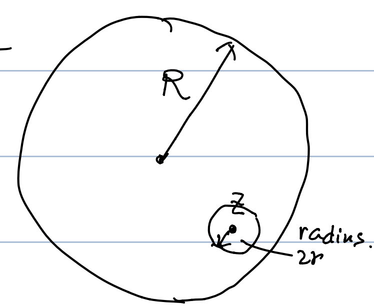
claim: $\limsup_{n\to\infty} |a_{n+m}|^{\frac{1}{n}} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$ $(m>0)$

idea: $\limsup_{n\to\infty} \left[|a_{n+m}|^{\frac{1}{n+m}}\right]^{\frac{n+m}{n}} \overset{!}{=} \limsup_{n\to\infty} |a_{n+m}|^{\frac{1}{n+m}}$ $\quad \frac{n+m}{n} \to 1.$

$= \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}.$

$\left(\limsup_{n\to\infty} c_n^{\,d_n} = \limsup_{n\to\infty} c_n \quad \text{if } d_n \to 1.\right)$

(2). Want prove $f(z)$ has complex derivative for any z, s.t. $|z|<R$. Assume r is small enough, that $D_{2r}(z) \subset D_R(0)$. We want to show, $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $\forall |h| < \delta$, we have



$$\left| \frac{f(z+h) - f(z)}{h} - g(z) \right| < 3\varepsilon.$$

$$g(z) = \sum_{n=1}^{N} a_n \cdot n \cdot z^{n-1} + \sum_{n=N+1}^{\infty} a_n \cdot n \cdot z^{n-1}.$$

$$=: \quad g^{(N)}(z) \quad + \quad \tilde{g}^{(N)}(z).$$

$\uparrow$ tail

$$G(z,h) = \frac{f(z+h)-f(z)}{h} = \sum_{n=0}^{\infty} \frac{a_n \cdot (z+h)^n - a_n \cdot z^n}{h}.$$

$$= \sum_{n=0}^{N} a_n \cdot \frac{(z+h)^n - z^n}{h} + \sum_{n=N+1}^{\infty} a_n \cdot \frac{(z+h)^n - z^n}{h}.$$

$$= \quad G^{(N)}(z,h) \quad + \quad \tilde{G}^{(N)}(z,h)$$

$\uparrow$ tail

$$|G(z,h) - g(z)| = |G^{(N)}(z,h) - g^{(N)}(z) + \tilde{G}^{(N)}(z,h) - \tilde{g}^{(N)}(z)|.$$

$$\leq |G^{(N)}(z,h) - g^{(N)}(z)| + \underline{|\tilde{G}^{(N)}(z,h)|} + |\tilde{g}^{(N)}(z)|.$$

(1) There exist N_1 such that, $\forall N > N_1$,

$$|\tilde{g}^{(N)}(z)| < \varepsilon.$$

(This is because the series $\sum_{n=1}^{\infty} a_n \cdot n \cdot z^{n-1}$. converge. hence the tail can be made as small as one wants.).

(2). $$\tilde{G}^{(N)}(z,h) = \sum_{n=N+1}^{\infty} a_n \cdot \frac{(z+h)^n - z^n}{h}.$$

$$(z+h)^n = \underbrace{z^n + \binom{n}{1} \cdot z^{n-1} \cdot h + \binom{n}{2} z^{n-2} \cdot h^2 + \cdots + h^n}_{n+1 \text{ terms.}}$$

$$\frac{(z+h)^n - z^n}{h} = \binom{n}{1} \cdot z^{n-1} + \underline{\binom{n}{2}} z^{n-2} \cdot h + \cdots + \underline{h^{n-1}}. \quad (n \text{ terms})$$

$$\leq n \, (|z| + r)^n.$$

trouble $\binom{n}{k}$ can be large, though

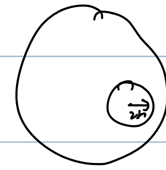
with some work, it might be cured by h^k.

$$a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}\cdot b + a^{n-3}\cdot b^2 + \cdots + b^{n-1}\right).$$

thus.

$$(z+h)^n - z^n = h\cdot\left((z+h)^{n-1} + (z+h)^{n-2}\cdot z + \cdots + z^{n-1}\right).$$

$$\left|\frac{(z+h)^n - z^n}{h}\right| = \left|(z+h)^{n-1} + (z+h)^{n-2}\cdot z + \cdots + z^{n-1}\right|.$$



if $|h| < r$, then. $|z|$ and $|z+h| < \underline{R-r}$

$$\leq n\cdot \underline{(R-r)}^{n-1}.$$

$$|\tilde{G}^{(N)}(z,h)| < \sum_{n=N+1}^{\infty} a_n\cdot n\cdot (R-r)^{n-1} \qquad \forall\ |h| < r.$$

again, $\exists N_2$, s.t. $\forall N > N_2$, $|\tilde{G}^{(N)}(z,h)| < \varepsilon$.

(3). Now for the main part. $\left|G^{(N)}(z,h) - g^{(N)}(z)\right|$.

Let's fix an N, $N > N_1$, $N > N_2$. then.

$\because$ the polynomial $\sum_{n=0}^{N} a_n\cdot z^n$ has derivative $\underbrace{\sum_{n=0}^{N} a_n\cdot n\cdot z^{n-1}}_{g^{(N)}(z)}$.

$$\therefore\ G^{(N)}(z,h) = \frac{\sum_{n=0}^{N} a_n (z+h)^n - \sum_{n=0}^{N} a_n\cdot z^n}{h} \longrightarrow g^{(N)}(z) \quad \text{as } h\to 0.$$

in other words, $\exists\ \delta_N > 0$, s.t. $\forall\ |h| < \delta_N$. we have.

$$\left|G^{(N)}(z,h) - g^{(N)}(z)\right| < \varepsilon.$$

So we have found a δ, s.t. $\forall\ |h| < \delta$.

$$|G(z,h) - g(z)| < 3\varepsilon.$$

$\Rightarrow$ complex derivative of $f(z)$ exist at point z #.

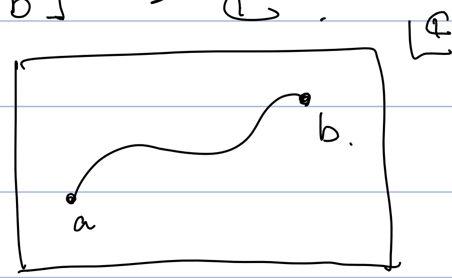
Cor: A power series is infinitely complex differentiable in its disc of convergence. And all derivatives $f^{(n)}(z)$ are holomorphic, and equals the termwise differentiation.

$$f^{(k)}(z) = \sum_{n=0}^{\infty} a_n \left(\frac{d}{dz}\right)^k (z^n)$$

3. Integration along curves.

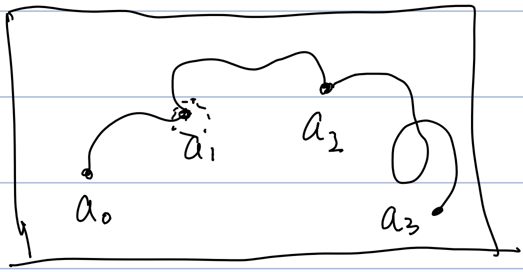
- what is a curve (in $\mathbb{C}$) ?
- parametrized smooth curve:

a (C^1) smooth map $\gamma : [a,b] \to \mathbb{C}$.



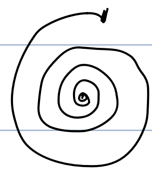
we also need a technical condition:
$\gamma'(t) \neq 0$ for all $t \in [a,b]$.

- parametrized piecewise smooth curve.



why we need $\gamma'(t) \neq 0$?

To avoid some weird curves :



$$\gamma : [0,1] \to \mathbb{C}$$

$$\gamma(t) = e^{-\frac{1}{t} + \frac{i}{t}}$$

indeed $|\gamma(t)| = e^{-\frac{1}{t}} \to 0$ as $t \to 0^+$

$\arg(\gamma(t)) = {}^{i}/_{t}$

$$\lim_{t \to 0} |\gamma'(t)| \sim \underset{\sim}{\frac{1}{t^2}} \cdot \underline{e^{-\frac{1}{t}}} \to 0 \quad \text{as } t \to 0^+.$$

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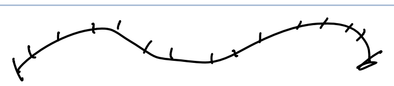
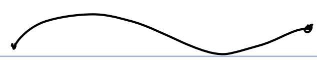
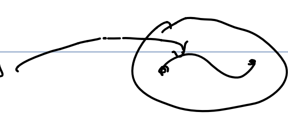
- (unparametrized) curve = consider all <u>equivalent</u> parametrization of the curve.

param curve



a b



- Integration on a curve : [0,1]  Ω

- $f : \Omega \to \mathbb{C}$ hol'c function
- $\gamma : [0,1] \to \Omega$. smooth curve.

$$\int_\gamma f\, dz := \int_0^1 \underline{f(z(t)) \cdot \gamma'(t)} \cdot dt$$

integration with complex integrand.

To show next time. , if we use an equivalent param
$t': [0,1] \to [a,b]$.

· $\tilde{\gamma} : [a,b] \to \Omega$, s.t. $\exists$ $t'(t)$,

$$\tilde{\gamma}(t'(t)) = \gamma(t).$$

then, we get the same result for integral.