Math 185 Take home Final Dec 15-17, 2020

Name: ______________________________

- You have 48 hours to complete the exam: from Dec 15, 12noon(PST) to Dec 17, 12noon.
- Please upload your solution in a single pdf to gradescope.
- Please provides all intermediate steps for calculation problems and justifications for proof based problems.
- This is a open-book exam, you can use your textbooks, lecture notes and homework solutions. You can only quote results contained in the above sources.
- No calculator should be used. No searches on internet are allowed.
- The final should reflect your own understanding. No discussion or collaboration of any sorts are allowed.
- If you have question during the exam, you may contact me use zoom direct message or via email.

Good Luck!

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

We use $\mathbb{D} = \{z : |z| < 1\}$ for the open unit disk, $C = \partial\mathbb{D}$ for its boundary, i.e the unit circle, and $\widehat{\mathbb{C}}$ for the extended complex plane $\mathbb{C} \cup \{\infty\}$.

1. (10 points, 2 points each)

   (1) What is the definition of a holomorphic function? What is the Cauchy-Riemann condition?

   (2) What is the definition of radius of convergence for a power series?

   (3) What is the maximum principle for holomorphic function?

   (4) What is normal family and the Arzela-Ascoli theorem?

   (5) What is the Riemann mapping theorem?

2. (10 points, 2 points each) True or False. Please provide your reasoning.

   (1) If f is a holomorhpic function on $\mathbb{D}$ and f vanishes on infinitely many points $z_1, z_2, \cdots$ in $\mathbb{D}$, then f has to be zero.

   (2) If f is a holomorphic function on $\mathbb{D}$, and $|f|$ is constant, then f has to be constant as well.

   (3) For any $a \in \mathbb{D}$, the function $f(z) = (a - z)/(1 - \overline{a}z)$ is a bijection from $\mathbb{D}$ to $\mathbb{D}$.

   (4) Let f_n be a sequence of holomorphic functions on $\mathbb{D}$, such that f_n converges uniformly on every compact subset of $\mathbb{D}$, then

   $$f(z) = \lim_{n\to\infty} f_n(z), \quad z \in \mathbb{D}$$

   is holomorphic on $\mathbb{D}$.

   (5) If f is a holomorphic function on a neighborhood of the unit circle C, then there always exist holomorphic functions f_1 on $\{|z| \le 1\}$ and f_2 on $\{|z| \ge 1\}$, such that $f(z) = f_1(z) - f_2(z)$ for all $|z| = 1$.

3. Let $f = \frac{1}{z-1} + \frac{1}{z-2}$.

   (1) (5 points) Compute the Taylor series expansion centered at $z = 0$. What is the radius of convergence?

   (2) (5 points) Compute the Laurent series expansion of f on the annulus $1 < |z| < 2$. i.e. find the coefficients b_n, such that

   $$f(z) = \sum_{n=-\infty}^{\infty} b_n z^n, \qquad \text{for all } 1 < |z| < 2$$

4. Compute the following integrals.

   (1) (3 points)

   $$\int_{|z|=2} \frac{e^z}{z(z-1)} dz$$

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(2) (3 points)

$$\int_{-\infty}^{+\infty} \frac{1}{(x+i)(x+2i)} dx$$

(3) (4 points)

$$\int_{|z|=1} \frac{1}{\sin(1/z)} dz$$

5. (10 points) If $Q(z)$ is a polynomial with distinct roots $\alpha_1, \cdots, \alpha_n$, and $P(z)$ is a polynomial with degree less than n, then show that we have partial fraction

$$\frac{P(z)}{Q(z)} = \sum_{i=1}^{n} \frac{P(\alpha_i)}{Q'(\alpha_i)(z-\alpha_i)}$$

6. (10 points) (a) (5 points) Let f be a holomorphic function defined in a neighborhood of $\overline{\mathbb{D}}$, such that $|f(z)| = 1$ for $|z| = 1$ and $f(z) \neq 0$ for $|z| < 1$. Show that f is a constant.

   (b) (5 points) Let f be a holomorphic function defined in a neighborhood of $\overline{\mathbb{D}}$, such that $|f(z)| = 1$ for $|z| = 1$. Show that f can be extended to a rational function on $\mathbb{C}$ and there are no roots of f outside $\mathbb{D}$.

7. (10 points) If the power series $\sum_n a_n z^n$ has radius of convergence $R_1 > 0$ and $\sum_n b_n z^n$ has radius of convergences $R_2 > 0$, show that the radius of convergence of the power series $\sum_n a_n b_n z^n$ is at least $R_1 R_2$.

8. (10 points) In each of the following cases, write down an entire function $f(z)$ such that,

   (1) (5 points) f has simple zeros exactly at $z = n^2$ with $n = 1, 2, 3, \cdots$.

   (2) (5 points) f has simple zeros exactly at $z = n$ with $n = 1, 2, 3, \cdots$.

9. (10 points) Normal Family for holomorphic functions.

   (1) (5 points) Let $\Omega = \{|z| < 1/2\}$, and let $\mathcal{F}$ be a family of holomorphic function on Ω, consisting of polynomials of the form

$$f(z) = (z-a_1)\cdots(z-a_n), \quad |a_i| < 1/2, \quad \forall i = 1, \cdots, n.$$

   Is $\mathcal{F}$ a normal family on Ω? Justify your anwer.

   (2) (5 points) Let $\Omega = \{|z| < 1\}$, and let $\mathcal{F}$ be a family of holomorphic function on Ω, consisting of

$$f(z) = \frac{1}{z-a}, \quad |a| > 1, \quad |z| < 1.$$

   Is $\mathcal{F}$ a normal family on Ω? Justify your anwer.

10. (10 points) Let f be a holomorphic function defined on the upper half-plane $\mathbb{H} = \{z : Im(z) > 0\}$, such that for any $z \in \mathbb{H}$, we have $Im(f(z)) \geq 0$. Show that for any $z, z_0 \in \mathbb{H}$, we have

$$\frac{|f(z) - f(z_0)|}{|f(z) - \overline{f(z_0)}|} \leq \frac{|z - z_0|}{|z - \overline{z_0}|}.$$

Hint: For any $a \in \mathbb{H}$, the map

$$z \mapsto \frac{z - a}{z - \overline{a}}$$

is a biholomorphic map from $\mathbb{H}$ to $\mathbb{D}$. Then use Schwarz lemma.

#2 (1) False. Only if $\{z_i\}$ has a limit point in $\mathbb{D}$, theorem 4.8 (Stein) would apply, and force $f=0$.

Ex : $f(z) = \sin\left(\frac{1}{z-1}\right)$. has roots at $\left\{\frac{1}{1+n\pi} : n \in \mathbb{Z}\right\}$. infinitely many in $\mathbb{D}$

(2) True. We did it in Stein Ch 1. Ex 13.

(3) True. Stein. Ch 1. Ex 7

(4) True. Stein Thm 5.2 (pg 53)

(5) True. Let $\varepsilon > 0$ be small enough, such that f is hol'c on $\{1-\varepsilon < |z| < 1+\varepsilon\}$. Then, by the proof of existence of Laurent expansion in Ahlfors, we get the desired f_1, f_2. Explicitly,

$$f_1(z) = \frac{1}{2\pi i} \int_{|w|=1+\varepsilon/2} \frac{f(w)}{w-z}\, dw \qquad \forall\ |z| \leq 1$$

$$f_2(z) = \frac{1}{2\pi i} \int_{|w|=1-\varepsilon/2} \frac{f(w)}{w-z}\, dw \qquad \forall\ |z| \geq 1.$$

#3. $f = \frac{1}{z-1} + \frac{1}{z-2}$.

(1) For $|z| < 1$, we have Taylor expansions.

$$\frac{1}{z-1} = -\frac{1}{1-z} = -(1+z+z^2+z^3+\cdots) = \sum_{n=0}^{\infty} -z^n$$

$$\frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-(z/2)} = -\frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right) = \sum_{n=0}^{\infty} -\left(\frac{1}{2}\right)^{n+1} \cdot z^n$$

Thus

$$f(z) = \sum_{n=0}^{\infty} -\left(1+\left(\frac{1}{2}\right)^{n+1}\right) \cdot z^n. \qquad \forall.\ |z|<1.$$

(2) For $1<|z|<2$, we have.

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-1/z} = \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) = \sum_{n=-1}^{-\infty} z^n.$$

$$\frac{1}{z-2} = \sum_{n=0}^{\infty} -\left(\frac{1}{2}\right)^{n+1} \cdot z^n \qquad \text{as before.}$$

$$f(z) = \sum_{n=-1}^{-\infty} z^n + \sum_{n=0}^{\infty} -\left(\frac{1}{2}\right)^{n+1} \cdot z^n.$$

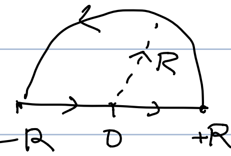
#4 (1) $$\int_{|z|=2} \frac{e^z}{z(z-1)}\,dz = 2\pi i \left(\text{Residues of } \frac{e^z}{z(z-1)} \text{ inside } |z|<2\right)$$

$$\operatorname{Res}_{z=0} \frac{e^z}{z(z-1)} = \frac{e^0}{(0-1)} = -1$$

$$\operatorname{Res}_{z=1} \frac{e^z}{z(z-1)} = \frac{e^1}{1} = e$$

$\therefore$ integral $= 2\pi i(-1+e)$.

(2) $I = \int_{-\infty}^{+\infty} \frac{1}{(x+i)(x+2i)} \cdot dx$

we may consider the following integral contour $C_R =$ 

$$I_R = \int_{C_R} \frac{1}{(z+i)(z+2i)} dz = \underbrace{\int_{-R}^{R} \frac{1}{(x+i)(x+2i)} \cdot dx}_{I_{R,1}} + \underbrace{\int_{\substack{|z|=R \\ \mathrm{Im}\, z > 0}} \frac{1}{(z+i)(z+2i)} dz}_{I_{R,2}}.$$

$|I_{R,2}| \leq C \cdot \frac{1}{R^2} \cdot R \longrightarrow 0$ as $R \to \infty$

$I_R = 2\pi i \cdot \left(\text{Residue of } \frac{1}{(z+i)(z+2i)} \text{ inside } C_R \right)$

$= 0$ since the poles are at $z = -i$, $z = -2i$

Hence,

$$I. = \lim_{R\to\infty} I_{R,1} = \lim_{R\to\infty} (I_R - I_{R,2}) = 0.$$

(3) $I = \oint_{|z|=1} \frac{1}{\sin(1/z)} dz = -\oint_{|w|=1} \frac{1}{\sin w} d\left(\frac{1}{w}\right) = \oint_{|w|=1} \frac{1}{\sin w} \cdot \frac{dw}{w^2}.$

$\uparrow$ $w = \frac{1}{z}$

There is only one pole inside $|w| = 1$, at $w = 0$. We compute the residue there: near $w = 0$

$$\frac{1}{w^2 \cdot \sin w} = \frac{1}{w^2 \left(w - \frac{w^3}{3!} + \cdots\right)} = \frac{1}{w^3 \left(1 - \frac{w^2}{6} + \cdots\right)}$$

$$= \frac{1}{w^3} \left(1 + \left(\frac{w^2}{6} + \cdots\right) + \left(\frac{w^2}{6} + \cdots\right)^2 + \cdots \right) = \frac{1}{w^3} \left(1 + \frac{w^2}{6} + \cdots \right)$$

$$= \frac{1}{w^3} + \frac{1}{6} \frac{1}{w}. \quad \Rightarrow \operatorname{Res}_{w=0} \frac{1}{w^2 \sin w} = \frac{1}{6}.$$

$$\therefore \quad I = 2\pi i \cdot \left(\operatorname{Res}_{w=0} \frac{1}{w^2 \cdot \sin} \right) = \pi i / 3.$$

#5. Both sides has the same locations of poles and residues, hence the difference is a holomorphic function. Since the difference vanishes at ∞, by maximum principle, (or Liouville theorem), difference has to be zero.

#6. (a) Consider $h(z) = 1/f$, then $|h(z)| = 1$ for $|z|=1$. By maximum principle, $|h(z)| \leq 1$ if $|z| \leq 1$. This forces $|f| = 1$ $\forall |z| \leq 1$, hence f is a constant on $\mathbb{D}$

(b) Let $\alpha_1, \cdots, \alpha_N$ be the zeros of f inside $\mathbb{D}$ (repeated with multiplicity), then we define

$$F(z) = \prod_{i=1}^{N} \frac{\alpha_i - z}{1 - \bar{\alpha}_i \cdot z} \qquad \forall |z| \leq 1.$$

If $|z| = 1$, then $|F(z)| = \prod_{i=1}^{N} \left| \frac{\alpha_i - z}{1 - \bar{\alpha}_i z} \right| = 1.$

Hence, $g(z) = f(z) / F(z)$ is a holomorphic function on $\overline{\mathbb{D}}$, with no zeros inside $\mathbb{D}$, and $|g(z)| = 1$ if $|z|=1$, hence is a constant $e^{i\theta}$ by (a). Thus.

$$f(z) = e^{i\theta} \cdot \prod_{i=1}^{N} \frac{\alpha_i - z}{1 - \bar{\alpha}_i z} \qquad \forall |z| \leq 1.$$

The right-hand side defines a meromorphic extension of $f(z)$. And we see that it has no zeros outside $\mathbb{D}$.

(b) alternatively, we can use a version of the Schwarz reflection principle: define

$$F(z) = \begin{cases} f(z) & |z| \le 1 \\ \overline{1/f(1/\bar{z})} & |z| \ge 1. \end{cases}$$

Indeed, if $z = e^{i\theta}$, $1/\bar{z} = 1/e^{-i\theta} = e^{i\theta} = z$, hence if $f(e^{i\theta}) = e^{i\varphi}$, then

$$\overline{1/f(e^{i\theta})} = e^{i\varphi}$$ as well. Hence $\forall |z|=1$.

$$f(z) = \overline{1/f(1/\bar{z})}.$$

One can check that, $\forall |z| \ge 1$,

$$\partial_{\bar{z}}(F(z)) = \overline{\partial_z(1/f(1/\bar{z}))} = 0$$

And by an application of Morera theorem, one get $F(z)$ is hol'c near $|z| = 1$. This extension sends each zero of $f(z)$ at $z = \alpha$ in $\mathbb{D}$ to pole at $z = 1/\bar{\alpha}$.

#7 Suffice to prove that, for any $0 < r < R_1 R_2$, the sum $\sum_{n=0}^{\infty} |a_n b_n| r^n$ is convergent. Let $r = r_1 \cdot r_2$, for $r_1 \in (0, R_1)$, $r_2 \in (0, R_2)$. Then.

$$\sum_n |a_n||b_n| r^n = \sum_{n=0}^{\infty} (|a_n| r_1^n) \cdot (|b_n| r_2^n)$$

Since $\sum_n |a_n| r_1^n$ and $\sum_n b_n r_2^n$ are convergent, thus.

for any $\varepsilon > 0$, there exists $N > 0$, such that.

$\sum_{n=N+1}^{\infty} |a_n| r_1^n < \sqrt{\varepsilon}$, $\quad \sum_{n=N+1}^{\infty} |b_n| r_2^n < \sqrt{\varepsilon}$.

Then. $\quad \sum_{n=N+1}^{\infty} |a_n| \cdot |b_n| \cdot r_1^n \cdot r_2^n$

$$\leq \left(\sum_{n=N+1}^{\infty} |a_n| \cdot r_1^n \right) \cdot \left(\sum_{n=N+1}^{\infty} |b_n| \cdot r_2^n \right)$$

$$\leq \varepsilon.$$

Hence $\sum_{n=0}^{\infty} |a_n| \cdot |b_n| \cdot r^n$ is convergent.

Alternatively, we may use

$$\limsup_{n \to \infty} |a_n b_n|^{\frac{1}{n}} \leq \limsup_n |a_n|^{\frac{1}{n}} \limsup_n |b_n|^{\frac{1}{n}}$$

$$= 1/R_1 R_2$$

to quickly conclude that the radius of convergemes is at least $R_1 R_2$.

#8. (a) $\quad f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2}\right)$

(b) $\quad f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) \cdot e^{z/n}$

See Ahlfors and Hw #9 for why the product converges.

#1 Recall the Montel theorem about criteria of Normal family for holomorphic function:
A family $\mathcal{F}$ of hol'c functions on Ω is a normal family if and only if over any compact subset $K \subset \Omega$, $\mathcal{F}$ is uniformly bounded.

(1). $\forall f \in \mathcal{F}$, $\forall z \in \{|z| < 1/2\}$,

$$|f(z)| = |(z-\alpha_1) \cdots (z-\alpha_n)| < 1.$$

Hence $\mathcal{F}$ is uniformly bounded on Ω, thus is a normal family

(2). $\forall K \subset \mathbb{D}$ compact subset, there exists $r<1$, such that $K \subset \{|z| < r\}$.

Then $\forall f \in \mathcal{F}$, i.e. $f(z) = \frac{1}{z-a}$ for $a \in \overline{\mathbb{D}}^c$.
and $\forall z \in K$, we have

$$|f(z)| = \frac{1}{|z-a|} \leq \frac{1}{\operatorname{dist}(K, \mathbb{D}^c)} < \frac{1}{1-r}$$

Hence $\mathcal{F}$ is uniformly bounded, and $\mathcal{F}$ is a normal family.

#10 (I forgot to say, f is non-constant.)

(1) first, we claim that $\forall z_0 \in \mathbb{H}$, $f(z_0) \in \mathbb{H}$ instead of $f(z_0) \in \overline{\mathbb{H}}$. This is guranteed by open mapping theorem.

· Let $\psi_a(z) = \dfrac{z-a}{z-\bar{a}}$, $\forall a \in \mathbb{H}$. $z \in \mathbb{H}$.

then $|z-a| < |z-\bar{a}|$, $|\psi_a(z)| < 1$.

Then ψ_a is a biholomorphism from $\mathbb{H}$ to $\mathbb{D}$.

· Consider the holomorphic map

$$F(z) = \psi_{f(z_0)} \circ f \circ \psi_{z_0}^{-1}(z) \; : \; \mathbb{D} \to \mathbb{D}.$$

$$F(0) = \psi_{f(z_0)} \cdot f \cdot \psi_{z_0}^{-1}(0) = \psi_{f(z_0)}(f(z_0)) = 0$$

Hence we may apply Schwarz Lemma, get

$$|F(w)| \le |w| \qquad \forall w \in \mathbb{D}.$$

$\forall z \in \mathbb{H}$, let $w = \psi_{z_0}(z)$, then

$$\left|\psi_{f(z_0)}(f(z))\right| \le |\psi_{z_0}(z)|$$

as desired.