Math 185 Take home Final Dec 15-17, 2020

Name: ______________________________

- You have 48 hours to complete the exam: from Dec 15, 12noon(PST) to Dec 17, 12noon.
- Please upload your solution in a single pdf to gradescope.
- Please provides all intermediate steps for calculation problems and justifications for proof based problems.
- This is a open-book exam, you can use your textbooks, lecture notes and homework solutions. You can only quote results contained in the above sources.
- No calculator should be used. No searches on internet are allowed.
- The final should reflect your own understanding. No discussion or collaboration of any sorts are allowed.
- If you have question during the exam, you may contact me use zoom direct message or via email.

Good Luck!

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

We use $\mathbb{D} = \{z : |z| < 1\}$ for the open unit disk, $C = \partial\mathbb{D}$ for its boundary, i.e the unit circle, and $\widehat{\mathbb{C}}$ for the extended complex plane $\mathbb{C} \cup \{\infty\}$.

1. (10 points, 2 points each)

   (1) What is the definition of a holomorphic function? What is the Cauchy-Riemann condition?

   (2) What is the definition of radius of convergence for a power series?

   (3) What is the maximum principle for holomorphic function?

   (4) What is normal family and the Arzela-Ascoli theorem?

   (5) What is the Riemann mapping theorem?

2. (10 points, 2 points each) True or False. Please provide your reasoning.

   (1) If f is a holomorhpic function on $\mathbb{D}$ and f vanishes on infinitely many points $z_1, z_2, \cdots$ in $\mathbb{D}$, then f has to be zero.

   (2) If f is a holomorphic function on $\mathbb{D}$, and $|f|$ is constant, then f has to be constant as well.

   (3) For any $a \in \mathbb{D}$, the function $f(z) = (a - z)/(1 - \overline{a}z)$ is a bijection from $\mathbb{D}$ to $\mathbb{D}$.

   (4) Let f_n be a sequence of holomorphic functions on $\mathbb{D}$, such that f_n converges uniformly on every compact subset of $\mathbb{D}$, then

   $$f(z) = \lim_{n\to\infty} f_n(z), \quad z \in \mathbb{D}$$

   is holomorphic on $\mathbb{D}$.

   (5) If f is a holomorphic function on a neighborhood of the unit circle C, then there always exist holomorphic functions f_1 on $\{|z| \leq 1\}$ and f_2 on $\{|z| \geq 1\}$, such that $f(z) = f_1(z) - f_2(z)$ for all $|z| = 1$.

3. Let $f = \frac{1}{z-1} + \frac{1}{z-2}$.

   (1) (5 points) Compute the Taylor series expansion centered at $z = 0$. What is the radius of convergence?

   (2) (5 points) Compute the Laurent series expansion of f on the annulus $1 < |z| < 2$. i.e. find the coefficients b_n, such that

   $$f(z) = \sum_{n=-\infty}^{\infty} b_n z^n, \quad \text{for all } 1 < |z| < 2$$

4. Compute the following integrals.

   (1) (3 points)

   $$\int_{|z|=2} \frac{e^z}{z(z-1)} dz$$

(2) (3 points)

$$\int_{-\infty}^{+\infty} \frac{1}{(x+i)(x+2i)} dx$$

(3) (4 points)

$$\int_{|z|=1} \frac{1}{\sin(1/z)} dz$$

5. (10 points) If $Q(z)$ is a polynomial with distinct roots $\alpha_1, \cdots, \alpha_n$, and $P(z)$ is a polynomial with degree less than n, then show that we have partial fraction

$$\frac{P(z)}{Q(z)} = \sum_{i=1}^{n} \frac{P(\alpha_i)}{Q'(\alpha_i)(z-\alpha_i)}$$

6. (10 points) (a) (5 points) Let f be a holomorphic function defined in a neighborhood of $\overline{\mathbb{D}}$, such that $|f(z)| = 1$ for $|z| = 1$ and $f(z) \neq 0$ for $|z| < 1$. Show that f is a constant.

   (b) (5 points) Let f be a holomorphic function defined in a neighborhood of $\overline{\mathbb{D}}$, such that $|f(z)| = 1$ for $|z| = 1$. Show that f can be extended to a rational function on $\mathbb{C}$ and there are no roots of f outside $\mathbb{D}$.

7. (10 points) If the power series $\sum_n a_n z^n$ has radius of convergence $R_1 > 0$ and $\sum_n b_n z^n$ has radius of convergences $R_2 > 0$, show that the radius of convergence of the power series $\sum_n a_n b_n z^n$ is at least $R_1 R_2$.

8. (10 points) In each of the following cases, write down an entire function $f(z)$ such that,

   (1) (5 points) f has simple zeros exactly at $z = n^2$ with $n = 1, 2, 3, \cdots$.

   (2) (5 points) f has simple zeros exactly at $z = n$ with $n = 1, 2, 3, \cdots$.

9. (10 points) Normal Family for holomorphic functions.

   (1) (5 points) Let $\Omega = \{|z| < 1/2\}$, and let $\mathcal{F}$ be a family of holomorphic function on Ω, consisting of polynomials of the form

$$f(z) = (z-a_1)\cdots(z-a_n), \quad |a_i| < 1/2, \quad \forall i = 1, \cdots, n.$$

   Is $\mathcal{F}$ a normal family on Ω? Justify your anwer.

   (2) (5 points) Let $\Omega = \{|z| < 1\}$, and let $\mathcal{F}$ be a family of holomorphic function on Ω, consisting of

$$f(z) = \frac{1}{z-a}, \quad |a| > 1, \quad |z| < 1.$$

   Is $\mathcal{F}$ a normal family on Ω? Justify your anwer.

10. (10 points) Let f be a holomorphic function defined on the upper half-plane $\mathbb{H} = \{z : Im(z) > 0\}$, such that for any $z \in \mathbb{H}$, we have $Im(f(z)) \geq 0$. Show that for any $z, z_0 \in \mathbb{H}$, we have

$$\frac{|f(z) - f(z_0)|}{|f(z) - \overline{f(z_0)}|} \leq \frac{|z - z_0|}{|z - \overline{z_0}|}.$$

Hint: For any $a \in \mathbb{H}$, the map

$$z \mapsto \frac{z - a}{z - \overline{a}}$$

is a biholomorphic map from $\mathbb{H}$ to $\mathbb{D}$. Then use Schwarz lemma.