Math 185 First Midterm October 6, 2020

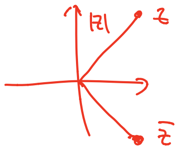
Name: ____

- You have 80 minutes to complete the exam, 9:40-11:00am.
- Please write your name and page number on every page that you submit. The submission deadline is at 11:10am.
- This is a open-book exam, you can use your textbook and notes.
- You may only use the results covered in class so far, including results in the lecture note and results in Stein up to Chapter 2.
- If you have question during the exam, you may contact me in zoom,
- Please write neatly. Answers which are illegible for the reader cannot be given credit.

Good Luck!

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	20	
9	10	
Total	100	

$\left|\frac{z}{\bar z}\right| = \frac{|z|}{|\bar z|}$



$|z|$

1. (10 points, 2 points each)

$|z|^2 = x^2+y^2 = \cdots = z\bar z$

   (1) Use z and $\bar z$ to express $\mathrm{Re}(z), \mathrm{Im}(z), |z|^2$.

   $z = re^{i\theta}$, $\bar z = r \cdot e^{-i\theta}$

   (2) If $z = 2020 + 1006i$, then $|z/\bar z| =$?. 1

   (3) If $z = (1/2)e^{i\pi/3}$, then $1/\bar z =$? $2e^{i\pi/3}$

   $\bar z$ ×

   $|z|$ ×

   (4) Give an example of a continuous function $f : \mathbb{C} \to \mathbb{C}$, where f is holomorphic at 0 but no other point in $\mathbb{C}$. (no justification needed) $f = \bar z^2$. $f = |z|^2$.

   (5) State the Cauchy-Riemann criterion for a function f to be holomorphic.

   · $f = u + iv$, · u, v continuously differentiable and $\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases}$

   u, v C^1, $\partial_x u = \partial_y v$, $\partial_x v = -\partial_y u$

2. (10 points, 2 points each) Let f be a holomorphic function on the unit open disk $\mathbb{D}$. Determine whether the following statements is true or false. No justification needed.

   $f(z) = \sum_{n=0}^{\infty} a_n \cdot z^n$ $\quad f_n(z) = \sum_{i=0}^{n} a_i \cdot z^i$.

   T (1) There exists a sequence of polynomials f_n, such that for any compact set $K \subset \mathbb{D}$, f_n converges to f uniformly on K ✓ Power series expansion partial sum.

   F (2) If f vanishes at infinitely many points in $\mathbb{D}$, then f is zero. ✗ $\sin\left(\frac{1}{z-1}\right)$

   T (3) If there is a point $z_0 \in \mathbb{D}$, such that $f^{(n)}(z_0) = 0$ for all $n = 0, 1, 2 \cdots$, then $f = 0$. ✓ use power series expansion at z_0. $\Rightarrow$ f vanishes in a nbhd of z_0.

   F (4) Let γ be a closed piecewise smooth curve in $\mathbb{D}$, possibly with self-intersection, then it is possible that $\int_\gamma f(z)dz \neq 0$. ✗

   F (5) If $f(0) = 0$ and $f'(0) = 1$, then $f(z) = z$. × $f(z) = 0 + z + z^2 + z^3 + \frac{1}{4}z^4$.

3. (10 points) Let $\Omega \subset \mathbb{C}$ be a region (open and connected subset), and $f : \Omega \to \mathbb{C}$ a holomorphic function. Suppose there is a line $L \subset \Omega$, such that f is constant on L. Show that f is constant in Ω. Let that constant be c, then $f - c$ vanishes on L. Apply Thm 4.8 in Stein.

4. (10 point) Show that the function $f : \mathbb{C}\backslash[0, 1] \to \mathbb{C}$

$$f(z) = \int_0^1 \frac{1}{z - t}dt$$

is holomorphic in $\mathbb{C}\backslash[0, 1]$, and its derivative is

$$f'(z) = \int_0^1 \frac{-1}{(z - t)^2}dt.$$

$$\frac{f(z+h) - f(z)}{h} = \frac{1}{h}\int_0^1 \left(\frac{1}{z+h-t} - \frac{1}{z-t}\right)dt = \int_0^1 \left(\frac{-1}{(z+h-t)(z-t)}\right)dt \to \int_0^1 \frac{-1}{(z-t)^2}dt$$

as $h \to 0$ $\because$ uniform convergence of integrand as $h \to 0$.

Hence complex derivative exist

(Hint: Use difference quotient to compute the derivative. Do not pass differentiation under the integral sign without justification.)

dominated convergence thm: if: $f_n \to f$. almost everywhere and if $|f_n| < |g|$, and $\int |g|\,dx$ exist. then $\lim_{n\to\infty} \int f_n\,dx = \int \lim_{n\to\infty} f_n\,dx$

5. (10 point) Let $K \subset \mathbb{C}$ be a compact set, and $f : K \to \mathbb{C}$ is a continous function. Is it always possible to find a sequence of polynomials $f_n(z)$, such that f_n converges to f uniformly on K? If yes, give a reference. If no, give your reason and a counter example. No. say $K = \{|z| = 1\}$,

   $\subset \mathbb{D}$

   ① f is continuous and $K = \mathbb{D}$. is it true ?

   ② if f is hol'c but K^c is not connected, then impossible.

6. (10 point) If $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function, and there is a constant $C > 0$, such that $|f(z)| < C(1 + |z|)$. Show that $f(z) = a + bz$ for some $a, b \in \mathbb{C}$.

f has power series expansion, valid for all $z \in \mathbb{C}$

$f(z) = \sum_{n=0}^{\infty} a_n \cdot z^n \qquad a_n = \frac{f^{(n)}(0)}{n!}$. Suffice to show

$f^{(n)}(0) = 0 \quad \forall\, n \geq 2$.

By Cauchy estimate. (Cor 4.3 p48). $\|f\|_{C_R}$

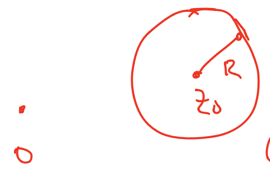
C_R R 0

$|f^{(n)}(0)| \leq \frac{n!}{R^n} C(1+R) \quad \forall R > 0$.

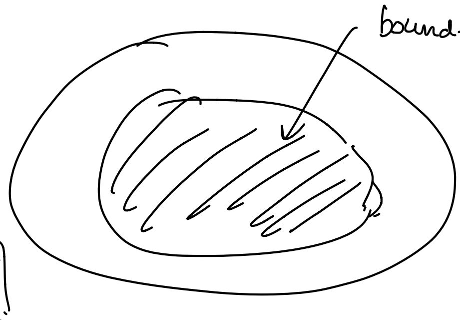
Hence, Let $R \to \infty$, we see $|f^{(n)}(0)| = 0$. $\forall\, n \geq 2$.

alternatively, consider $f^{(2)}(z_0)$ $\forall z_0$.

$|f^{(2)}(z_0)| \leq \frac{2! \cdot C(1+|z_0|+R)}{R^2} \to 0$ as $R \to \infty$

$C_R(z_0)$ 

Liouville thm:

$\mathbb{C} \xrightarrow{f}$  bounded. 

Here. $\mathbb{C} \xrightarrow{f}$ 

$\because |f(z) - z_0| > r$

$\therefore \dfrac{1}{|f(z)-z_0|} < \dfrac{1}{r}$

(cf. Picard thm).

7. (10 point) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Assume that there exists a point $z_0 \in \mathbb{C}$ and an open neighborhood $D_r(z_0)$, such that $f(\mathbb{C}) \cap D_r(z_0) = \emptyset$. Show that f is a constant function.

Apply Liouville theorem (Cor 4.5, P50) to $\dfrac{1}{f(z)-z_0}$.

8. (20 points, 10 points each) Evaluate the following contour integrals.

(1)

$$\oint_{|z|=1} \frac{(z+2)(z+3)}{z(z+4)(z+5)} dz = ?$$

Let $F(z) = \dfrac{(z+2)(z+3)}{(z+4)(z+5)}$. $F(z)$ hol'c for $\overline{\mathbb{D}}$.

$$\int_{|z|=1} \frac{F(z)}{z}\, dz = 2\pi i \cdot F(0) = 2\pi i \cdot \frac{3}{10} = \frac{3}{5}\pi i$$

By Cauchy integral formula.

(2) For any real number $a > 1$, evaluate

$$\oint_{|z|=1} \frac{1}{|z-a|^2} |dz|$$

$|dz| = R \cdot d\theta = R \cdot \dfrac{dz}{iz} = \dfrac{dz}{iz}$

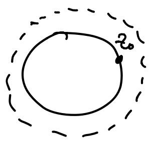
$$\oint \frac{1}{(z-a)(\bar{z}-a)} \frac{dz}{iz} = \oint \frac{1}{(z-a)(\frac{1}{z}-a)} \frac{dz}{iz}$$

$$= \oint \frac{-i}{(z-a)(1-az)} dz = \oint \frac{+i\, a^{-1}}{(z-a)(z-a^{-1})} dz$$

$$= \frac{2\pi \cdot a^{-1}}{a-a^{-1}} = \frac{2\pi}{a^2-1}$$

9. (10 point) Let $g(z)$ be a holomorphic function on an open neighborhood of $\mathbb{D}$, $z_0 \in \mathbb{C}$ with $|z_0| = 1$. Let $f(z) = \frac{g(z)}{z-z_0}$. Consider the follow~~e~~ power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \quad (*)$$



Show that

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

① radius of convergence of $(*)$ is 1

$\not\Rightarrow \displaystyle\lim_{n\to 1} \frac{|a_n|}{|a_{n+1}|} = 1$

(Hint: write $g(z) = g(z_0) + (z - z_0)h(z)$.)

Pf: $\cdot\ f(z) = \dfrac{g(z_0)}{z-z_0} + h(z)$

- consider power series expansion of $h(z)$ at $z=0$

$$h(z) = \sum_{n=0}^{\infty} b_n z^n.$$

It's radius of convergence is $R > 1$, i.e.

$$\limsup_{n\to\infty} |b_n|^{\frac{1}{n}} = \frac{1}{R} < 1$$

Let $\varepsilon > 0$ be small enough, such that $\frac{1}{R} < 1-\varepsilon < 1$

$\exists N_0$, s.t $\forall n > N_0$, $|b_n|^{\frac{1}{n}} < (1-\varepsilon)$, i.e. $|b_n| < (1-\varepsilon)^n$.

- $$\frac{g(z_0)}{z-z_0} = \frac{-g(z_0)}{z_0} \frac{1}{1-z/z_0} = \frac{-g(z_0)}{z_0}\left(1 + \left(\frac{z}{z_0}\right) + \left(\frac{z}{z_0}\right)^2 + \cdots\right)$$

$$= \sum_{n=0}^{\infty} c_n \cdot z^n, \qquad c_n = \frac{-g(z_0)}{z_0} \cdot \frac{1}{z_0^n}$$

3

$\displaystyle\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = z_0$

- $$\lim_{n\to\infty} \frac{|b_n|}{|c_n|} \le \lim_{n\to\infty} \frac{(1-\varepsilon)^n}{|g(z_0)|} = 0.$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{b_n + c_n}{b_{n+1} + c_{n+1}} = \lim_{n\to\infty} \frac{\frac{b_n}{c_n} + 1}{\frac{b_{n+1}}{c_{n+1}} + 1} \cdot \lim \frac{c_n}{c_{n+1}} = z_0$$