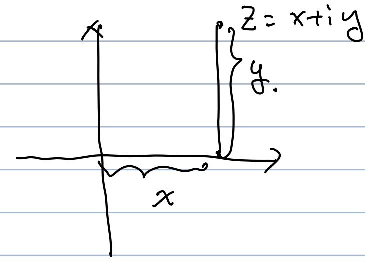
Discussed:

1. Complex Number $\mathbb{C} \simeq \mathbb{R}^2$

- $z = x + iy$ $\qquad x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$.



complex conjugate. $\bar{z} = x - iy$.

$$x = \frac{z+\bar{z}}{2}, \qquad y = \frac{z-\bar{z}}{2i}$$

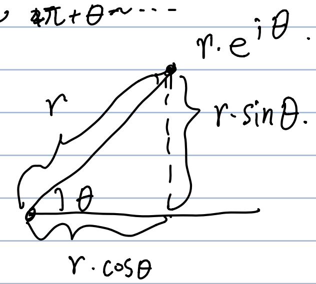
useful for addition and subtraction.

- $z = r \cdot e^{i\theta}$. $\qquad r \geq 0$, $\theta \in \mathbb{R} \bmod 2\pi\mathbb{Z}$.

$\theta \sim 2\pi + \theta \sim 4\pi + \theta \sim \cdots$

$$(r \cdot e^{i\theta}) \cdot (r' \cdot e^{i\theta'}) = (rr') \cdot e^{i(\theta+\theta')}$$

$$(r \cdot e^{i\theta}) / (r' \cdot e^{i\theta'}) = (r/r') \cdot e^{i(\theta-\theta')}$$

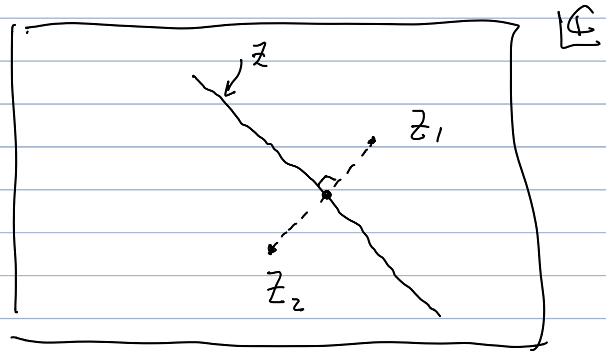


r: modulus of z. (absolute value of z).
θ: argument. (phase).

- $$\begin{cases} x = r \cdot \cos\theta \\ y = r \cdot \sin\theta \end{cases}$$

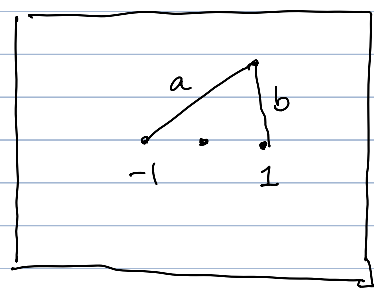
Ex (from Stein):

(1) $|z - z_1| = |z - z_2|$ $\qquad z_1, z_2 \in \mathbb{C}$



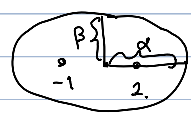
(1b)

$$|z-1| + |z+1| = 3$$



$a+b=3$

answer: ellipse

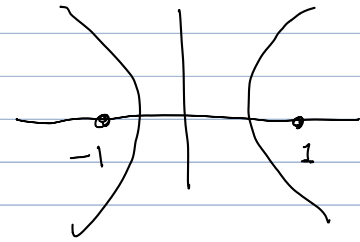


$z = x + iy$.

$$|z-1| = |x+iy-1| = |(x-1)+iy| = \sqrt{(x-1)^2+y^2}$$

$$\sqrt{(x-1)^2+y^2} + \sqrt{(x+1)^2+y^2} = 3. \dashrightarrow \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1.$$

(1c) $|z-1| - |z+1| = 3$



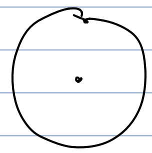
$z = x+iy = r\cdot e^{i\theta}$

$\bar{z} = x-iy = r\cdot e^{-i\theta}$

$z\bar{z} = x^2-(iy)^2 = x^2 - i^2 y^2$
$= x^2+y^2 = |z|^2$
$= r\cdot e^{i\theta}\cdot r\cdot e^{-i\theta} = r^2$

(2) $1/z = \bar{z}$

$\hookrightarrow 1 = z\cdot\bar{z} = |z|^2$



$|z|=1.$

$\mathbb{C} \to \mathbb{R}$

(3). $\overbrace{\mathrm{Re}(az+b)}^{z} > 0$, $a, b, \in \mathbb{C}$.

$a = a_1 + i a_2$, $b = b_1 + i b_2$.

$z = x+iy.$

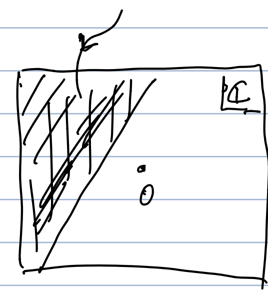
$az+b = (a_1+ia_2)(x+iy) + (b_1+ib_2)$

$= a_1 x - a_2 y + b_1 + i(\cdots\cdots)$

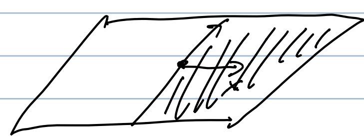
$\mathrm{Re}(az+b) = a_1 x - a_2 y + b_1$

$a_1 x - a_2 y + b_1 > 0$

$\{z \in \mathbb{C} \mid \mathrm{Re}(az+b) > 0\}$



$\mathrm{Re}(z) > 0$
$\{x > 0\}$



(2) Holomorphic function = "exist complex derivative".

example	non-example (non-hol'c)
$f(z) = z^n$	$\bar{z}$.
$f(z) = 1/z$ (hol'c if $z \neq 0$)	$\lvert z\rvert^2$
e^z, $\sin z$	$\arg(z)$

ex: $\overset{f(z)}{\leftarrow}$ $|z|$ is not holomorphic (at any point. $z_0 \in \mathbb{C}$)

e.g. if $z_0 = 0$. , we need to check.

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = \lim_{h\to 0} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

cplx

cplx

$(h = |h| \cdot e^{i\theta_h})$.

$$= \lim_{h\to 0} \cdot e^{-i\theta_h}$$

does not exist. Because if h approach 0 from different directions., the result $e^{-i\theta_h}$ are not the same, hence the limit doesn't exist.

- Remark: Having complex derivative at z_0 means.

$$f(z_0+h) = \underbrace{f(z_0)}_{\text{constant term}} + \underbrace{f'(z_0)\cdot h}_{\text{linear term.}} + \underbrace{\cdots}_{\text{small.}}$$

for $|h| < \varepsilon$

"open connected subset"

- we will see, if f is hol'c on a domain Ω, then.
 f has all the complex derives, ($f'(z)$ is hol'c, $f''(z)$ is hol'c ...)

$$\frac{f(z_0+h)-f(z_0)}{h} \longrightarrow f'(z_0) \quad \text{as } h\to 0.$$

$$\frac{f(z_0+h)-f(z_0)}{h} = f'(z_0) + R(z_0,h).$$

remainder term

and $R(z_0,h) \to 0$ as $h\to 0$.

$$f(z_0+h) = f(z_0) + h\cdot f'(z_0) + \underbrace{h\cdot R(z_0,h)}_{\text{Small.}}$$

- Power Series:

- a series ('in number).: it is an infinite sum.

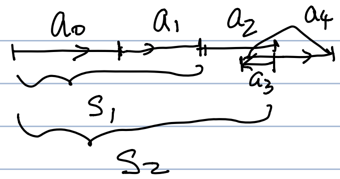
$$a_0 + a_1 + a_2 + \cdots\cdots = \sum_{n=0}^{\infty} a_n. = \sum_n a_n.$$

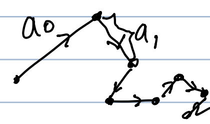
- a series converge., if the partial sum.

$$S_n = a_0 + a_1 + \cdots + a_n$$

has a limit: i.e. $\lim_{n\to\infty} S_n$ exist

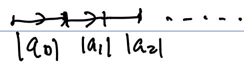
- A necessary condition is $\underline{a_n \to 0}$, as $n \to \infty$ ($\Leftrightarrow |a_n| \to 0$)





(not sufficient. i.e. ($\sum_n \frac{1}{n}$ is not convergent). $\qquad \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

- Absolute Convergence: $\sum_n a_n$ absolutely converge, iff $\sum_n |a_n|$ converge.

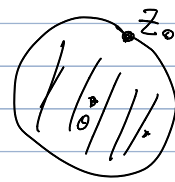


- Power series $\quad \sum_{n=0}^{\infty} a_n \cdot z^n$

- we say a power series converge at a point $z = z_0$, if

$$\sum_{n=0}^{\infty} a_n z_0^n \quad \text{converge.}$$

- Fact: If $\sum_{n=0}^{\infty} a_n \cdot z^n$ is absolutely convergent at $z = z_0$, then it is abs. conv. for all z, s.t. $|z| \le |z_0|$.



($\because \sum_{n=0}^{\infty} |a_n| \cdot |z|^n$ convergent.

if $|z| \le |z_0|$. then $|z|^n \le |z_0|^n$.

$\therefore |a_n| \cdot |z|^n \le |a_n| \cdot |z_0|^n$.

$\therefore \sum_{n=0}^{\infty} |a_n| \cdot |z|^n \le \sum_{n=0}^{\infty} |a_n| \cdot |z_0|^n < \infty$

Ex: $\cdot$ $e^z := 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\quad (0! = 1)$

a power series.

this ~~sum~~ series. is always convergent, for all $z \in \mathbb{C}$.

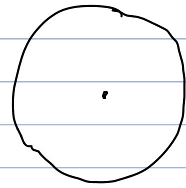
by ratio test: $\rho = \lim_{n\to\infty} \frac{|z^{n+1}/(n+1)!|}{|z^n / n!|} = \lim_{n\to\infty} \left| \frac{z}{n+1} \right| \to 0$

$\because \rho < 1 \quad \therefore$ by ratio test, $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is absolutely conv.

$\boxed{e^{x+iy} = e^x \cdot e^{iy} \quad \text{polar form of a cplx number}}$

• Radius of convergence (of a power series $\sum_{n=0}^{\infty} a_n \cdot z^n$).

$\infty \geq R \geq 0$, such. $\begin{cases} \text{if } |z| < R, & \text{then } \sum a_n z^n \text{ abs. conv.} \\ \text{if } |z| > R, & \text{then ---- is divergent.} \end{cases}$



Thm 2.5: the radius of convergence exists,

and is given by

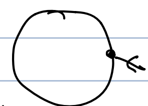
$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Ex: $1+z+z^2+z^3+\cdots = \sum_{n=0}^{\infty} 1 \cdot z^n$, $a_n = 1$.

$$\limsup_{n\to\infty} |1|^{\frac{1}{n}} = 1 = \frac{1}{R}. \quad \Rightarrow R = 1$$

if $|z|>1$, then $\sum z^n$ diverge., $\because |z|^n$ does not go to zero.

if $|z|<1$, $\sum z^n = \frac{1}{1-z}$.



$\frac{1}{1-z}$ diverges at $z=1$.

$\frac{1}{1-z}$ makes sense as long as $z \neq 1$

$\sum z^n$ only make sense for $|z|<1$.

Pf: • Assume for simplicity that

$$L = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \quad \text{is not } 0, \infty.$$

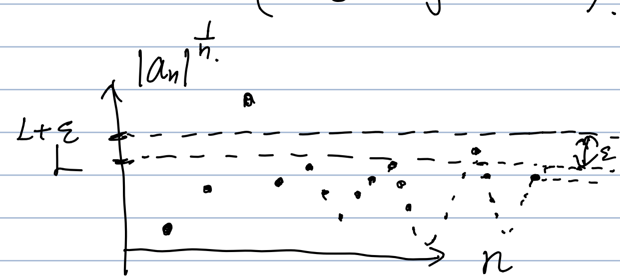
. $R = 1/L$.

(a). Need to show, if $|z| < R$, then. $\sum a_n z^n$ abs. conv.

$|z| \cdot L < 1$ ("ε" of room).

$\exists\, \varepsilon > 0$, small enough, such that.

$|z| \cdot (L+\varepsilon) < 1$.



Then, $\because \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = L$

$\exists N$ large enough, s.t. $n > N$.

$$|a_n|^{\frac{1}{n}} < L+\varepsilon.$$

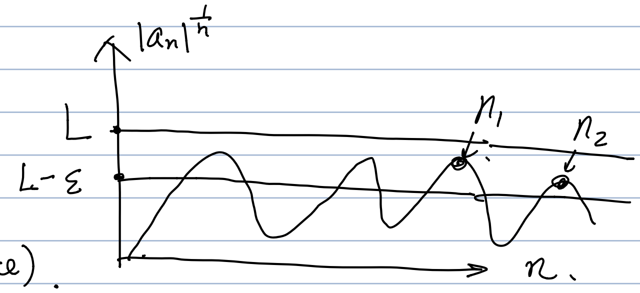
then $\sum_{n=N+1}^{\infty} |a_n| \cdot |z|^n < \sum_{n=N+1}^{\infty} (L+\varepsilon)^n \cdot |z|^n$

$= \sum_{n=N+1}^{\infty} \big(\underbrace{|z| \cdot (L+\varepsilon)}_{<1} \big)^n$ converges.

(b) if $|z| > R$, then. $|z| \cdot L > 1$. $\exists\, \varepsilon > 0$, small enough,

$|z| \cdot (L - \varepsilon) > 1$

$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = L$



$\exists$ an infinite occurrence (or subsequence).

$|a_{n_k}|^{\frac{1}{n_k}} > L - \varepsilon$.

so, for such $n \in \{ n_1, n_2, n_3, \cdots\cdots \}$.

$|a_n| \cdot |z|^n > (L-\varepsilon)^n \cdot |z|^n = \big(\underbrace{|z| \cdot (L-\varepsilon)}_{\text{a fix number} > 1} \big)^n \to \infty$

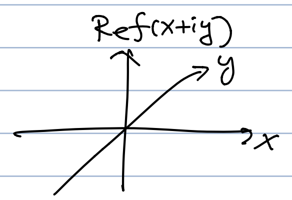
as $n \to \infty$ in this seq $\{ n_1, n_2, \cdots \}$.

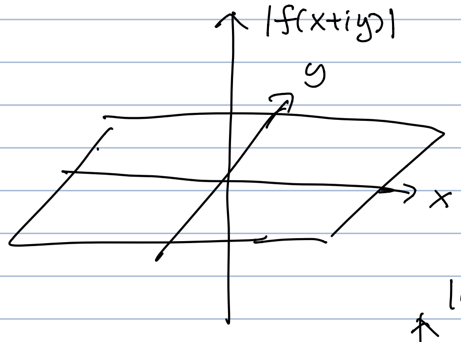
$|f(z)| : \mathbb{C} \to \mathbb{R}$

$\arg(z) : \mathbb{C} \to \mathbb{R}/2\pi\mathbb{Z}$.

$\mathrm{Re}(f(z))$

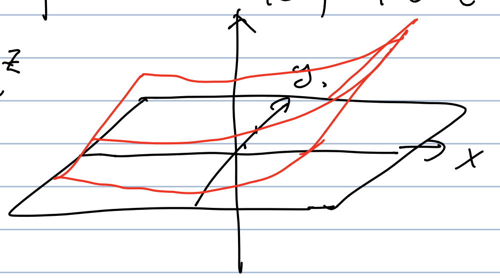
$\mathrm{Im}(f(z))$.



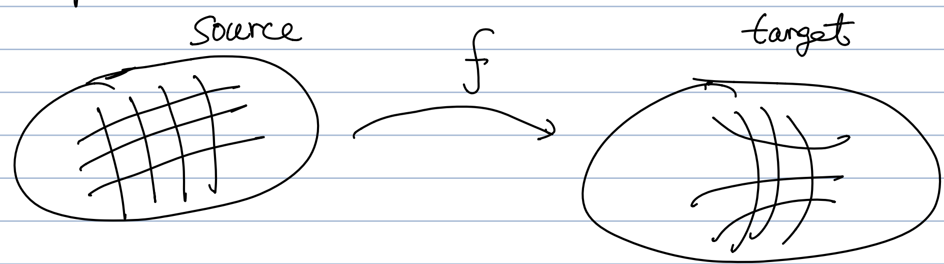


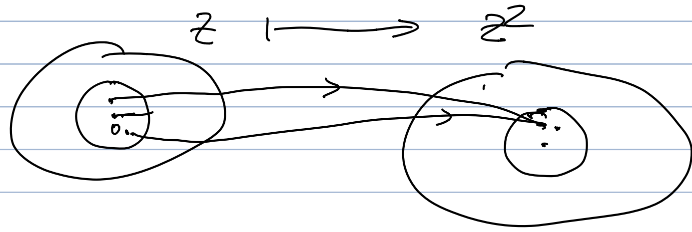
$|e^z| = |\, 1 \cdot e^x \cdot e^{iy} | = |e^x| \cdot |e^{iy}| = e^x$

e^z



another way:





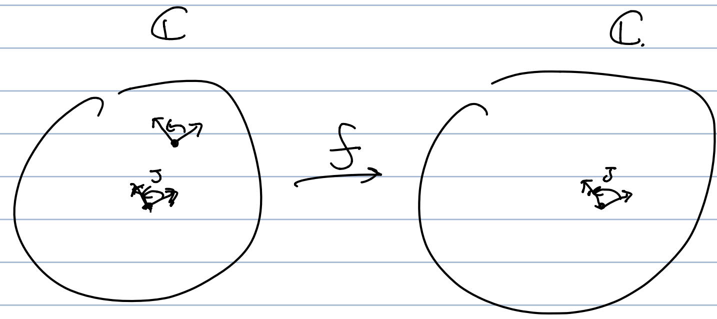
$f(z)$ hol'c. $\quad x+iy \mapsto u+iv$. $\quad u = \mathrm{Re}(f(z))$ $\quad v = \mathrm{Im}(f(z))$.

$u(x,y)$. $\quad v(x,y)$.

$\underline{f(z) \text{ is hol'c}} \iff \begin{cases} u, v \text{ differentiable.} \\ \underline{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}}, \quad \underline{\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}}. \end{cases}$

$\Updownarrow$

f is conformal.



$df \circ J = J \circ df$

↗ send forward the tangent vector.

↑ rotate 90°.