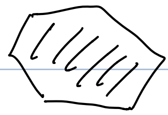
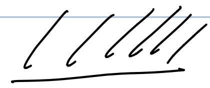
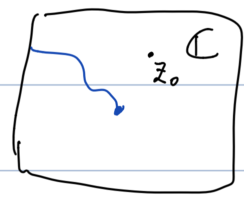
Final Goal : (1) Prove Riemann mapping Theorem.

· statement : Let $\Omega \subset \mathbb{C}$ be a simply connected region, then for any $z_0 \in \Omega$, there exists a unique "bi-holomorphic" map $f: \Omega \to \mathbb{D}$, such that $f(z_0) = 0$, $f'(z_0) > 0$.

· Ex: (1) $\Omega =$  interior of a polygon

(2) $\Omega =$  $\{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$.

(3) $\Omega = \mathbb{C} \setminus$ (blue) ray. closed set.



strategy : (1) construct a family $\mathcal{F}$ of functions $f: \Omega \to \mathbb{D}$ s.t. ① f is holomorphic, injective. (may not be surjective)

② $f(z_0) = 0$, $f'(z_0) > 0$. (or $|f'(z_0)|$).

(2) take the limit inside $\mathcal{F}$ to maximize $f'(z_0)$.

To show the limit exist, we need to introduce the notion of Normal family.

Ahlfors : · Normal family in Ch5
· Riemann mapping thm. Ch6

Stein : short cut like approach.

(2) Recall last time:

· Let $\Omega \subset \mathbb{C}$ region.

· S be a metric space. with dist fcn. $d(\cdot,\cdot)$

· $\operatorname{Map}(\Omega, S)$ continuous.

(e.g. $S = \mathbb{R}$, $\mathbb{C}$, $\hat{\mathbb{C}}$ or $\mathbb{R}^d$, or Riem mfd)

· we equip $\operatorname{Map}(\Omega, S)$ with a metric : $f, g: \Omega \to S$, $\rho(f,g)$ is the distance.

Construct ρ as follows:

$$\rho(f,g) = \sum_{k=1}^{\infty} 2^{-k} \cdot \delta_k(f,g).$$

where

$$\delta_k(f,g) = \sup_{z \in E_k} \underline{\delta}(f(z), g(z)),$$

$\cdots \subset E_k \subset E_{k+1} \subset \cdots$. compact subsets in Ω, $\Omega = \bigcup_k E_k$.

$\forall a, b \in S$ $\quad \delta(a,b) = \dfrac{d(a,b)}{1 + d(a,b)}$, such modification make $0 \le \delta < 1$

$\boxed{\text{Prop}}$: If $\{f_n\}$ is a seq of fcn : $\Omega \to S$, $f : \Omega \to S$.

TFAE:

(1) $f_n \to f$ in ρ - distance. (i.e. $\lim_{n\to\infty} \rho(f_n, f) = 0$)

$\Updownarrow$

(2) $f_n \to f$ uniformly in every compact subset in Ω.

(a.k.a. "locally uniformly".).

- Normal Family.
  - Let $\mathcal{F} \subset \mathrm{Map}(\Omega, S)$ be a family of fcn $\Omega \to S$.

- Notions of continuity :
  - (continuity) : A fcn $f : \Omega \to S$ is continuous if

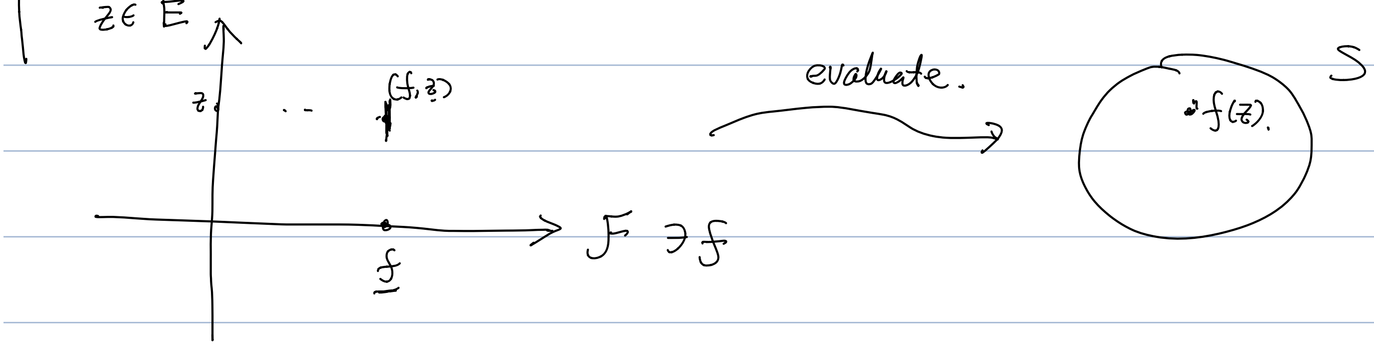
$\forall z_0 \in \Omega$, $\forall \varepsilon > 0$, $\exists \delta > 0$. (dep on f. z_0, ε). s.t.

$\forall z \in \Omega$, with $|z - z_0| < \delta$, we have $d(f(z), f(z_0)) < \varepsilon$.

- (uniform continuity) : A fcn $f : \Omega \to S$ is uniformly continuous on a subset $E \subset \Omega$, if $\forall \varepsilon > 0$, $\exists \delta > 0$ (dep. f, E, ε). s.t. $\forall z, z_0 \in E$. with $|z - z_0| < \delta$. we have $d(f(z), f(z_0)) < \varepsilon$.

☆ • (*equi-continuity*): Let $\mathcal{F}$ be a family of fcn, $\Omega \to S$.

We say $\mathcal{F}$ is equicontinuous on $E \subset \Omega$, if $\forall \varepsilon > 0$, $\exists \delta > 0$, (dep on $\mathcal{F}, E, \varepsilon$). s.t. $\forall f \in \mathcal{F}$, $\forall z, z_0 \in E$, s.t. $|z - z_0| < \delta$, we have. $d(f(z), f(z_0)) < \varepsilon$.



$$ev : \mathcal{F} \times E \to S$$
$$(f, z) \mapsto f(z).$$

equi-continuity means uniform continuity in both the f and z variable.

• Def (Normal Family). Let $\mathcal{F} \subset Map(\Omega, S)$. We say $\mathcal{F}$ is normal if and only if for every seq $\{f_n\}$ of functions in $\mathcal{F}$, it contains a subseq. that converges. uniformly on every compact subset in Ω.

(Note: the limit of the ~~seq~~ subseq is NOT required to be in $\mathcal{F}$).

Recall Thm (Bolzano - Weierstrass): a metric space is compact if and only if every seq has a convergent subsequence with limit in it.

Cor: Let, $\mathcal{F} \subset Map(\Omega, S)$, $Map(\Omega, S)$ is equipped with ρ dist.

Then $\mathcal{F}$ is a normal family $\Leftrightarrow$ the closure $\overline{\mathcal{F}}$ in $Map(\Omega, S)$ is compact.

(Terminology: if $A \subset X$, $\overline{A}$ is compact, we say A is relative cpt)

Thm (Arzela-Ascoli Thm).

Let $\mathcal{F} \subset Map(\Omega, S)$. continuous fcn. $\underset{\mathbb{C}}{\overset{\cap}{\Omega}} \to \underset{\text{metric space.}}{S}$

- $\mathcal{F}$ is normal $\Leftrightarrow$ $\begin{cases} \text{(i) } \mathcal{F} \text{ is equi-continuous on every cpt subset. } E \subset \Omega. \\ \text{(ii). for any } z \in \Omega, \text{ the values } f(z), \forall f \in \mathcal{F}, \text{ is contained in a compact subset of } S. \text{ (may dep on } z). \end{cases}$

Pf: $\Rightarrow$ (i). we need to show that $\forall E \subset \Omega$, cpt, $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $\forall f \in \mathcal{F}$, $\forall z_1, z_2 \in E$ with $|z_1 - z_2| < \delta$, $d(f(z_1), f(z_2)) < \varepsilon$.

$\mathcal{F}$ is normal $\Leftrightarrow$ $\overline{\mathcal{F}}$ is compact $\Leftrightarrow$ $\overline{\mathcal{F}}$ is complete and totally bounded.

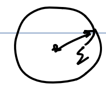
$\Rightarrow$ $\mathcal{F}$ totally bounded.

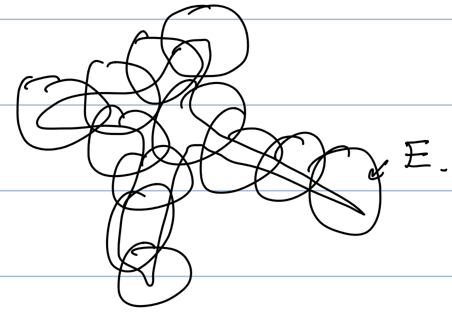
Def:

- a metric space is complete if every Cauchy seq has a limit.
- a subset E of a metric space S is totally bounded,

if for every $\varepsilon > 0$, there exist finitely many balls in S, of radius ε, that covers E.

(you may further insist that these balls have centers in E).





<u>Prop:</u>

$\mathcal{F}$ is <u>totally bounded</u> (as a subset in $\mathrm{Map}(\Omega, S)$ using ρ- distance).

$\Leftrightarrow$ $\forall$ compact subset $E \subset \Omega$, $\forall \varepsilon > 0$, $\exists f_1, \cdots, f_N \in \mathcal{F}$,

s.t. $\forall f \in \mathcal{F}$, $\exists i \in \{1, \cdots, N\}$. $d_E(f, f_i) < \varepsilon$.

$$d_E(f, g) = \sup_{z \in E} d(f(z), g(z))$$

Now return to the proof of "$\Rightarrow$ (i)"

- Using $\mathcal{F}$ is totally bounded, and given $E \subset \Omega$ cpt. $\varepsilon > 0$, we construct $(\varepsilon/3)$ "representative" $f_1, \cdots, f_N \in \mathcal{F}$. s.t. $\forall f \in \mathcal{F}$, $\exists i$, s.t. $\sup_{z \in E} d(f(z), f_i(z)) < \frac{\varepsilon}{3}$

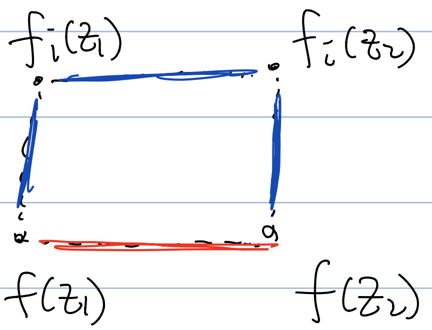
- Then focus on these $f_1, \cdots, f_N$ from E (cpt.) to S. hence we can get a δ, s.t. $\forall z_1, z_2 \in E$, $|z_1 - z_2| < \delta$. we have
$$d(f_i(z_1), f_i(z_2)) < \frac{\varepsilon}{3}, \quad \forall i \in \{1, \cdots, N\}.$$
(by uniform continuity of f_i on ~~these~~ E).

• with such δ chosen, we have, $\forall f \in \mathcal{F}$,

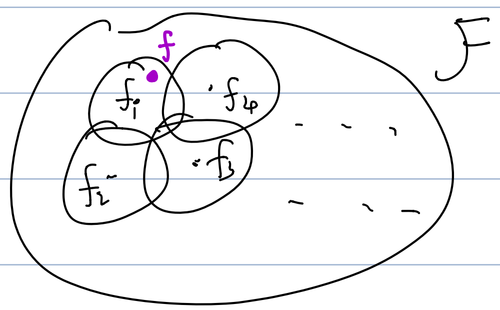
$\forall z_1, z_2 \in E$, $|z_1 - z_2| < \delta$, we apply the triangle ineq.

pick an $i \in \{1, \dots, N\}$ s.t. $d_E(f, f_i) < \frac{\varepsilon}{3}$.

$$d(f(z_1), f(z_2)) \le d(f(z_1), f_i(z_1)) + d(f_i(z_1), f_i(z_2)) + d(f_i(z_2), f(z_2))$$



$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \le \varepsilon.$$



"$\Rightarrow$ (ii)"

since $\overline{\mathcal{F}}$ is compact,

and

$$ev_{z_0} : \overline{\mathcal{F}} \to S, \quad f \mapsto f(z_0)$$

"sloppy part"

is $\boxed{\text{continuous}}$, Hence

$ev_{z_0}(\overline{\mathcal{F}})$ is compact.

$\therefore$ $ev_{z_0}(\mathcal{F}) \subset ev_{z_0}(\overline{\mathcal{F}})$ #.

$$ev_{z_0}(\mathcal{F}) = \{f(z_0) \mid f \in \mathcal{F}\}$$

evaluation map:

$$ev : Map(\Omega, S) \times \Omega \to S$$
$$(f, z) \mapsto f(z).$$

• $Map(\Omega, S)$ is equipped with the ρ- dist. metric.

• ev is continuous.

• $ev_{z_0} : Map(\Omega, S) \times \{z_0\} \to S$
$$f \mapsto f(z_0)$$

" $\Leftarrow$ (i) (ii) " Need to show, for all seq $\{f_n\}$ in $\mathcal{F}$, we can pick a subseq $\{f_{n_i}\}$ $n_1 < n_2 < n_3 < \cdots$ such that $\forall\, E \subset \Omega$ cpt, f_{n_i} converges uniformly on E.

We will use Cantor's diagonal argument.

- Pick a countable dense subset of Ω. $\mathbb{Q}^2$

$\hat{\Omega} = \{\xi_1, \xi_2, \xi_3, \cdots\}$ e.g. $\hat{\Omega} = \underbrace{(\mathbb{Q} + i\mathbb{Q})}_{\text{dense in } \mathbb{C}} \cap \Omega$.

- Construct an array of indices. n_{kj}

$$\begin{array}{cccccccc}
n_{11} & < & n_{12} & < & n_{13} & < & \cdots \\
n_{21} & < & n_{22} & < & n_{23} & < & \cdots \\
n_{31} & < & n_{32} & < & n_{33} & < & \cdots \\
\vdots & & \vdots & & & &
\end{array}$$

satisfying ① each row is contained in the previous row as a sequence

② $\lim\limits_{j \to \infty} \underline{f_{n_{kj}}(\xi_k)}$ exists.

construct it row by row, e.g. for $k=1$. we look at point ξ_1. we have the full seq $\{f_1(\xi_1), f_2(\xi_1), \cdots\}$ since it is contained in a cpt subset (ii), it sub converges. repeat the step, with the sub sequence obtained from step 1, and point $\xi_2, \cdots$.

· Let $n_i = n_{ii}$, thus. $\forall \xi_k$,

$\lim_{i\to\infty} f_{n_i}(\xi_k)$ exists.

we claim $\{f_{n_i}\}$ converges uniformly on E.

pf claim: $\forall \varepsilon > 0$, need find $N > 0$, s.t. $\forall i, j > N$.

$d_E(f_{n_i}, f_{n_j}) < \varepsilon$.

By (i), equicontinity of $\mathcal{F}$ on E, $\exists \delta > 0$, s.t.

$\forall z_1, z_2 \in E$, $|z_1 - z_2| < \delta$, $\forall f \in \mathcal{F}$, we have $d(f(z_1), f(z_2)) < \frac{\varepsilon}{3}$.

By compactness of E, we can cover E by open balls

$\{B_\delta(\xi_n)\}_{n=1,\cdots}$, pick a finite subcover, say.

$B_\delta(\underline{\xi_1}), \cdots, B_\delta(\underline{\xi_m})$. (after relabel the index).

We pick N large enough. s.t. $\forall i, j > N$. $\forall \underline{a} \in \{1, \cdots, m\}$

$d(f_{n_i}(\xi_a), f_{n_j}(\xi_a)) < \frac{\varepsilon}{3}$..

Hence, $\forall z \in E$, $\exists \xi_a$ s.t. $|z - \xi_a| < \delta$.

$\forall i, j > N$, we have.

$d(f_{n_i}(z), f_{n_j}(z)) < d(f_{n_i}(z), f_{n_i}(\xi_a))$

$+ d(f_{n_i}(\xi_a), f_{n_j}(\xi_a)) + d(f_{n_j}(\xi_a), f_{n_j}(z))$.

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \qquad \#.$$