Preparatory work:

1. Normal family for analytic function (Montel's thm)

Thm: Let $\mathcal{F}$ be a family of analytic fcn $: \Omega \to \mathbb{C}$.

then $\mathcal{F}$ is normal $\iff$ $\mathcal{F}$ is locally uniformly bounded.

(Ahlfors P224 Thm 15). i.e. $\forall E \subset \Omega$ cpt, $\exists K \subset \mathbb{C}$ bounded.

(s.t. $\forall f \in \mathcal{F}$, $f(E) \subset K$.) $\iff$ $ev(\mathcal{F} \times E)$ bounded

Pf: $\Rightarrow$ $\forall E \subset \Omega$ cpt, $ev: \underset{\text{cpt}}{\overline{\mathcal{F}}} \times \underset{\text{cpt}}{E} \to \mathbb{C}$ is a continuous fcn with compact domain

$\therefore K = ev(\overline{\mathcal{F}} \times E)$ is compact, $\therefore$ $f(E) = ev(\{f\} \times E) \subset \underline{K}$.

$\Leftarrow$ we use Arzela-Ascoli thm, and try to prove

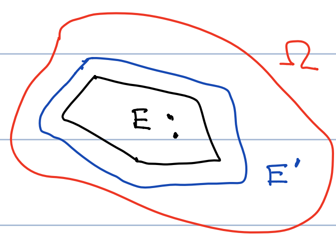
(i) $\mathcal{F}$ is equicontinuous on any compact subset $E \subset \Omega$.

(ii) $\forall z \in \Omega$, $ev(\mathcal{F} \times \{z\})$ is bounded.

(ii) is automatic by uniform boundedness, since we can take the compact subset $E = \{z\}$.

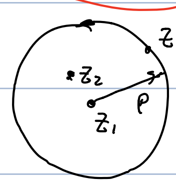
(i) follows from Cauchy integral formula. Let $E \subset \Omega$ be any compact subset. Let $\rho = \frac{1}{2}\operatorname{dist}(E, \Omega^c) > 0$. Then $\forall z \in E$. $D_\rho(z) \subset \Omega$, Let $E' = \{z \mid \operatorname{dist}(z, E) \le \rho\}$, then E' is compact.

Let $M = \sup_{z \in E'} \sup_{f \in \mathcal{F}} |f(z)|$, then by uniform boundedness, M is finite.



$\forall z_1, z_2 \in E$, if $|z_1 - z_2| < \underset{\sim}{\frac{\rho}{2}}$, then

$$f(z_1) - f(z_2) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_1} - \frac{f(z)}{z - z_2} \, dz$$



$|z - z_1| = \rho$

$|z - z_2| \geq \rho/2$.

$C = C_\rho(z_1)$

$$= \frac{1}{2\pi i} (z_1 - z_2) \int_C \frac{f(z)}{(z - z_1)(z - z_2)} \, dz$$

$$|f(z_1) - f(z_2)| \le \frac{1}{2\pi} |z_1 - z_2| \cdot 2\pi\rho \cdot M \frac{1}{\rho \cdot \rho/2} = \frac{2M}{\rho} |z_1 - z_2|.$$

$\forall \varepsilon > 0$, let $\delta > 0$ be small enough, such that

$$\delta < \frac{\rho}{2} \quad \text{and} \quad \frac{2M}{\rho} \cdot \delta < \varepsilon.$$

Then, $\forall f \in \mathcal{F}$, $\forall z_1, z_2 \in E$, with $|z_1 - z_2| < \delta$, we have.

$|f(z_1) - f(z_2)| < \varepsilon$. proving equicontinuity of $\mathcal{F}$ on E. #

2. Hurwitz thm. (Ahlfors P178)

If functions $f_n(z)$ are analytic and non-vanishing in a region Ω, and if $f_n(z) \to f(z)$, uniformly on every compact subset in Ω, then $f(z)$ is either ~~non zero~~ non-vanishing on Ω or identically zero.

Pf: Prove by contradition. Suppose $f \not\equiv 0$, and $\exists z_0 \in \Omega$, s.t. $f(z_0) = 0$. Then there is a small disk $B := B_r(z_0) \subset \Omega$, s.t. $f(z)\big|_{\bar{B} \setminus \{z_0\}} \neq 0$. Then $\frac{1}{2\pi i} \int_{\partial B} \frac{f'(z)}{f(z)} dz \neq 0$. However,

$$\lim_{n \to \infty} \frac{1}{2\pi i} \int_{\partial B} \frac{f_n'(z)}{f_n(z)} dz = \frac{1}{2\pi i} \int_{\partial B} \frac{f'(z)}{f(z)} dz. \neq 0$$

since $\frac{1}{f_n(z)} \to \frac{1}{f(z)}$ and $f_n'(z) \to f'(z)$, $\forall z \in \partial B$, and convergence is uniform on the compact set ∂B. Hence, contradiction with

$$\frac{1}{2\pi i} \int_{\partial B} \frac{f_n'(z)}{f_n(z)} dz = \# \text{ of zero of } f_n(z) \text{ inside } B = 0.$$

(This is essentially the same proof as Rouché theorem).

3. Schwarz Lemma (map from disk to disk).

Let $f: \mathbb{D} \to \mathbb{D}$ be hol'c with $f(0) = 0$. Then (Stein P218).

(1) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$

(2) If for some $z_0 \in \mathbb{D}$, $|f(z_0)| = |z_0|$, then f is a rotation

(3). $|f'(0)| \le 1$. And if equality holds, f is a rotation.

Pf: (1). Let $g(z) = f(z)/z$ for $z \in \mathbb{D}\setminus\{0\}$. Then near $z=0$,

use Taylor expansion, $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$, and $f(0)=0$

we have $a_0 = 0$, Hence $g(z) = a_1 + a_2 z + \cdots$ is hol'c near 0.

Thus $z=0$ is a removable singularity for $g(z)$.

If $|z| = r < 1$, then $|g(z)| = \left|\frac{f(z)}{r}\right| \le \frac{1}{r}$. Hence

$$\sup_{|z|<1} |g(z)| = \lim_{r\to 1} \sup_{|z|\le r} |g(z)| = \lim_{r\to 1} \sup_{|z|=r} |g(z)| \le \lim_{r\to 1} \frac{1}{r} = 1.$$

Thus $|f(z)| \le |z|$ $\forall z \in \mathbb{D}$.

(ii) If $\exists z_0 \in \mathbb{D}$ $|f(z_0)| = |z_0|$, then $|g(z_0)| = 1$. Thus by Maximum principle,

$g(z) = c$ in $\mathbb{D}$., for $|c| = 1$, i.e. $c = e^{i\theta}$.

Thus $f(z) = e^{i\theta} \cdot z$

(iii). $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} g(z) = g(0)$.

Since $|g(0)| \le 1$, and if $|g(0)| = |f'(0)| = 1$, then

by maximum principle, $g(z) = c$ as above.

• Def: An automorphism of a region Ω is holomorphic bijection

$f: \Omega \to \Omega$.

Ex: • identify map

$|\alpha|<1$

- $\Omega = \mathbb{D}$, $\alpha \in \mathbb{D}$. (Blaske factor)

$$F_\alpha(z) := \frac{\alpha - z}{1 - \bar{\alpha} z} \qquad \mathbb{D} \to \mathbb{D} \qquad \text{automorphism.}$$

s.t. $\underline{F_\alpha \circ F_\alpha = id.}$ $\qquad F_\alpha(0) = \alpha$

$F_\alpha(\alpha) = 0.$

Prop: If $f: \mathbb{D} \to \mathbb{D}$ is an automorphism, then f can be written uniquely as

$$f(z) = e^{i\theta} \frac{\alpha - z}{1 - \bar{\alpha} z},$$

for some $\theta \in [0, 2\pi)$, $\alpha \in \mathbb{D}$.

Pf: Let $\alpha = f^{-1}(0)$. then $\qquad 0 \xmapsto{F_\alpha} \alpha \xmapsto{f} 0$

$$g := f \circ F_\alpha : \mathbb{D} \to \mathbb{D}$$

is an automorphism, with $g(0) = 0$. By Schwarz Lemma. for g.

$\forall z \in \mathbb{D}$, $\underline{|g(z)| \le |z|}$. Since $g^{-1}: \mathbb{D} \to \mathbb{D}$ is also hol'c. thus.

$|g^{-1}(z)| \le |z| \iff |z| \le |g(z)|$.

Thus $|g(z)| = |z|$ $\forall z \in \mathbb{D}$, and $\overset{= e^{i\theta} \cdot z}{g(z)}$ is a rotation by Schwarz Lemma (iii). Thus $g \circ F_\alpha = f \circ F_\alpha \circ F_\alpha = f$,

$f(z) = e^{i\theta} \cdot \frac{\alpha - z}{1 - \bar{\alpha} z}$. #.

$\alpha = 0$

Cor: if $f: \mathbb{D} \to \mathbb{D}$ is an automorphism with $f(0) = 0$ then f is a rotation. If further $f'(0) \in \mathbb{R}_{>0}$ then $f'(0) = 1$ and f is the identity map.

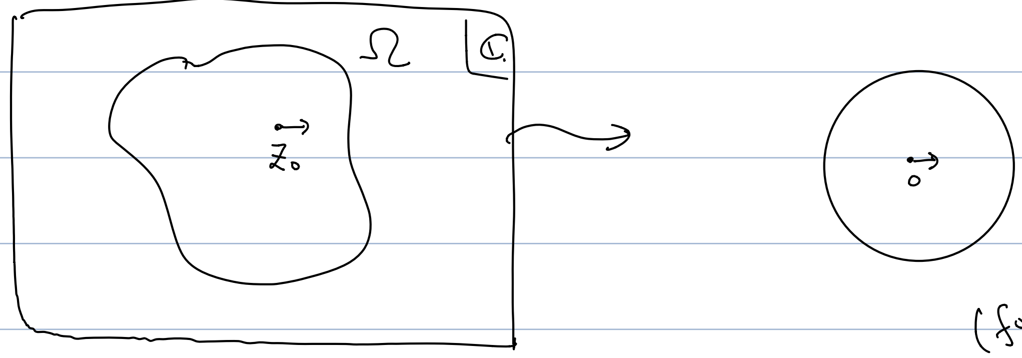
$f(z) = e^{i\theta} \cdot z$

$f'(0) > 0 \iff e^{i\theta} > 0$

Riemann Mapping Theorem:

Let $\Omega \subsetneq \mathbb{C}$ be a simply connected proper subset. Then for any $z_0 \in \Omega$, there exists a unique holomorphic map.

$f: \Omega \to \mathbb{D}$, s.t. $f(z_0)=0$, and $f'(z_0)>0$.



(follow Stein).

Pf: Let $\mathcal{F}$ be the family of holomorphic maps

$f: \Omega \to \mathbb{C}$ such that

① f is injective.

② $f(\Omega) \subset \mathbb{D}$

③ $f(z_0) = 0$,

Step (1): Show that $\mathcal{F}$ is non-empty.

By assumption, $\exists\, a \in \mathbb{C}$, $a \notin \Omega$.

$\boxed{\text{Easy case,}}$ if exists open nbhd $\mathcal{U}$ of a, such that $\mathcal{U} \subset \Omega^c$,

Then consider automorphism of $\hat{\mathbb{C}}$; $\Phi: z \longmapsto \frac{1}{z-a}$.

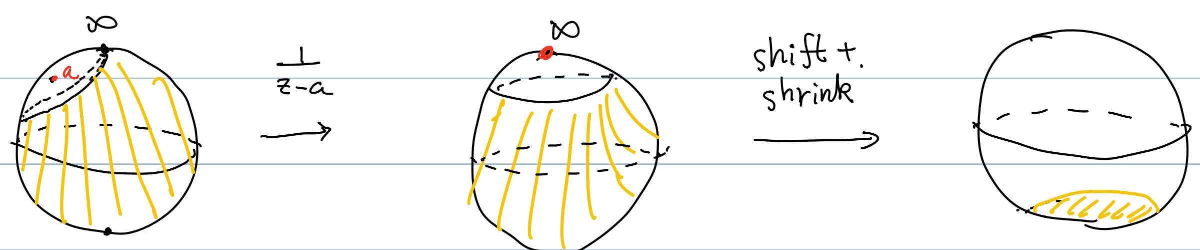
this will send $a \mapsto \infty$, $\Omega_1 = \Phi(\Omega) \subset \mathbb{C}$ still simply connected.

and Ω_1 is bounded, let $z_1 = \Phi(z_0)$. Then we consider the

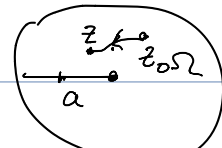
map $\Psi: \mathbb{C} \to \mathbb{C}$, $z \mapsto \frac{1}{R}(z - z_1)$. for some large enough R.

this will send Ω_1 to $\Omega_2 \subset \mathbb{D}$ and z_1 to 0. The composition

$\Psi \circ \Phi$ satisfies condition of $\mathcal{F}$.



Ω is unbounded. $\qquad$ Ω is bounded



$\boxed{\text{General case:}}$ Let $a \in \Omega^c$. Consider the function.

$$\underline{g(z)} = \int_{z_0}^{z} \frac{1}{w-a}\, dw \, ,$$ where integration is along any path in Ω from z_0 to z.

Since Ω is simply connected, (and $\frac{1}{w-a} \neq 0 \;\; \forall w \in \Omega$) the integral is well-defined.

Then $g'(z) = \frac{1}{z-a}$, $\quad g(z_0) = 0. \quad \Rightarrow \boxed{g(z) = \log\left(\frac{z-a}{z_0-a}\right)}$

with the branch of the logarithm suitably chosen. s.t. $g(z_0) = 0$.

Then $g : \Omega \to \mathbb{C}$ is injective, since $\exp \circ g : \; z \mapsto \frac{z-a}{z_0-a}$ is injective. Moreover $g(\Omega) \cap (g(\Omega) + 2\pi i \cdot n) = \phi$ for any $n \in \mathbb{Z} \setminus \{0\}$. Let $B = B_r(z_0) \subset \Omega$ be a ball, then.

$g(\Omega) \cap g(B) + 2\pi i = \phi$. Hence, the complement of $g(\Omega)$ contains an open subset $g(B) + 2\pi i$. We may apply the easy case to $\Omega' = g(\Omega)$ and $z_0' = g(z_0)$, and get a map $G : \Omega' \to \mathbb{D}$ with $G(z_0') = 0$. Then $G \circ g : \Omega \to \mathbb{D}$, $z_0 \mapsto 0$ satisfies the condition for $\mathcal{F}$. Hence $\mathcal{F}$ is non-empty.

Step ②: We may assume $\Omega \subset \mathbb{D}$ and $z_0 = 0$, since the general case can be reduced to this case.

By Montel's theorem, since $\mathcal{F}$ is uniformly bounded on Ω, $\mathcal{F}$ is a normal family.

Let $\underset{\sim}{S} = \sup_{f \in \mathcal{F}} |f'(0)|$. By Cauchy estimate, consider

a small closed disk $\overline{D_\varepsilon(0)} \subset \Omega$, we have $|f'(0)| < \frac{1}{\varepsilon}$, $\forall f \in \mathcal{F}$.

Hence $\underline{S < \infty}$. Let $\{f_n\}$ be a sequence in $\mathcal{F}$, s.t. $\lim_{n\to\infty} |f_n'(0)| = S$.

Then by normality of $\mathcal{F}$, $\{f_n\}$ has a subsequence that converges uniformly on every compact subset in Ω, to certain function f.

Thus. f is hol'c, and $f(\Omega) \subset \mathbb{D}$, $|f'(0)| = S$. Since

$\underline{id \in \mathcal{F}}$, $\therefore$ $\underline{S \geq 1}$.

We claim that f is injective. If not, then $\exists z_1, z_2 \in \Omega$.

such that $f(z_1) = f(z_2)$. Let $\{f_{n_k}\}$ be the aforementioned subseq.

Consider $g_k(z) = f_{n_k}(z) - \underline{f_{n_k}(z_1)}$, then $g_k(z_1) = 0$, g_k still injective on Ω

and $g_k(z) \to f(z) - f(z_1)$ locally uniformly on Ω. Since

$\{g_k(z)\}$ are non-vanishing on $\underline{\Omega \setminus \{z_1\}}$, hence By Hurwicz thm $f(z) - f(z_1)$ is

either non-vanishing on $\Omega \setminus \{z_1\}$ or constant. But $|f'(0)| \neq 0$

hence f is not a constant. Hence $f(z_2) - f(z_1) \neq 0$, and we get

a contradiction.

Thus, there is a function $f \in \mathcal{F}$ and $|f'(0)|$ achieves maximum.

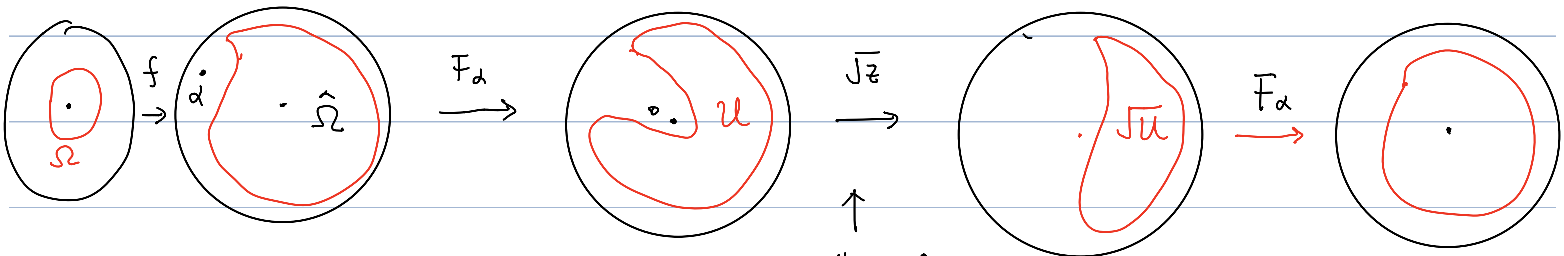
<u>Step (3)</u>: We prove $f: \Omega \to \mathbb{D}$ is surjective. If not,

then let $\alpha \in \mathbb{D} \setminus f(\Omega)$. Our aim is to explicitly construct

a "better" function: $\tilde{f}: \Omega \to \mathbb{D}$, with $\underline{|\tilde{f}'(0)|} > |f'(0)|$.

$$\tilde{f}: \Omega \xrightarrow{f} \mathbb{D} \xrightarrow{F_\alpha} \mathbb{D} \xrightarrow{\sqrt{z}} \mathbb{D} \xrightarrow{F_\alpha} \mathbb{D}$$

Let $\hat{\Omega} = f(\Omega)$, $\mathcal{U} = F_\alpha(\hat{\Omega})$, then $F_\alpha(\alpha) = 0$, and $0 \notin \mathcal{U}$.



well defined
since $\mathcal{U}$ is simply connected.

since $F_\alpha^2 = id$, thus.

$$\tilde{f} = F_\alpha \circ \sqrt{z} \circ F_\alpha \circ f$$

$$\Leftrightarrow \quad \underbrace{F_\alpha \circ z^2 \circ F_\alpha}_{\Phi} \circ \tilde{f} = f.$$

Φ maps $\mathbb{D}$ to $\mathbb{D}$, and is not injective, since z^2 is not.

Hence by the last part of Schwarz Lemma, $|\Phi'(0)| < 1$.

Thus.

$$f'(0) = \tilde{f}'(0) \cdot \Phi'(0)$$

$$|f'(0)| = |\tilde{f}'(0)| \cdot |\Phi'(0)| < |\tilde{f}'(0)|.$$

Contradicting with $|f'(0)|$ is maximum in $\mathcal{F}$. Hence f is surjective.

Finally, we may compose f with a rotation to achieve $f'(0) > 0$.

To show uniqueness, suffice to note that if f_1 and f_2 both satisfy the condition, then $f_{12} = f_2 \circ f_1^{-1} : \mathbb{D} \to \mathbb{D}$ is an automorphism that fixes 0 and $f_{12}'(0) > 0$, hence $f_{12} = id.$, i.e. $f_1 = f_2$. #.