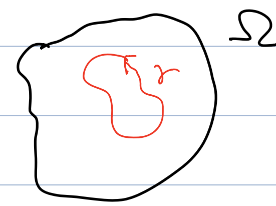
Today :

1. midterm 2

2. Ch 5 of Stein. Jensen.

1. Concepts & Thm :



- argument principle.

  f: meromorphic function on Ω.

  γ : simple closed curve.

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\,dz = \#\text{ zero } \underline{\text{inside } \gamma} - \#\text{ pole } \underline{\text{inside } \gamma}$$

- (3) If f is ~~an~~ a rational function on $\hat{\mathbb{C}}$, then does f has equal number of zero & pole? (including multiplicity)

	zero	pole.
Ex $f(z) = z$.	$z=0$.	$z=\infty$
$f(z) = \frac{1}{z-1}$	$z=\infty$	$z=1$.

$$f(z) = \frac{(z-a_1)\cdots(z-a_m)}{(z-z_1)\cdots(z-z_n)}$$

- if $n=m$. then ∞ is a regular point.

$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{z^n+\cdots}{z^n+\cdots} = 1.$$

zeros = $a_1, \cdots, a_m$

poles = $z_1, \cdots, z_n$.

$n=m$.

- if $n>m$, then ∞ is a zero of order $n-m$.

  $\#$ zero $= \#$ pole $= n$.

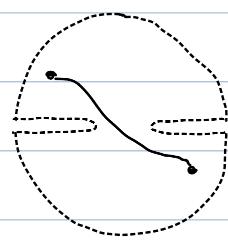
$\because$ let $w = 1/z$.
$$f(\tfrac{1}{w}) = \frac{(\frac{1}{w}-a_1)\cdots(\frac{1}{w}-a_m)}{(\frac{1}{w}-z_1)\cdots(\frac{1}{w}-z_n)}$$
$$= \frac{w^{n-m}(1-a_1w)\cdots(1-a_mw)}{(1-z_1w)\cdots(1-z_nw)}.$$

- similarly for poles.

(5): $\Omega = \mathbb{D} \setminus \{x+iy \mid y=0. \text{ and } x\in(-1,0.5]\cup[0.5,1)\}$

$\mathbb{D}$: open unit disk

$\Omega =$



it is simply connected.

2 (1). $f = \dfrac{(z+1)^2}{(z-1)^3}$ expand around $z=1$,

Let $u = z-1$, $f = \dfrac{(u+2)^2}{u^3} = \dfrac{u^2+4u+4}{u^3}$

$= \dfrac{1}{u} + \dfrac{4}{u^2} + \dfrac{4}{u^3}$

Residue at $u=0$ is 1.

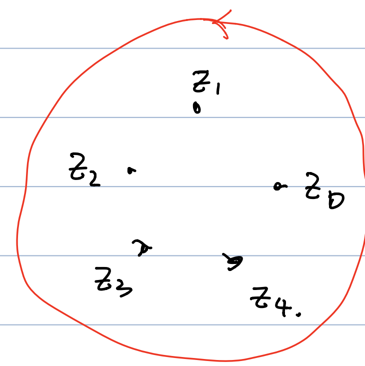
(2). $z^3 - 4z + 1$. $\rightarrow$ Rouché thm $\rightarrow$

$f(z) = -4z$, $g(z) = z^3 + 1$.

- $|f| > |g|$ on $\{|z| = 1\} = C$, then f and $f+g$ has same # of zero in C, which is $\underline{\underline{1}}$

- outside C. $\because$ total # zero $= 3$, $\therefore$ we have 2 zeros outside C.

(3). $$\oint_{\substack{|z|=2 \\ \text{CCW}}} \frac{1}{z^5 - 1}\, dz = 0$$



$z^5 - 1 = 0$ has 5 distinct roots $z_j = e^{\frac{2\pi i}{5} \cdot j}$, $j = 0, 1, \cdots, 4$.

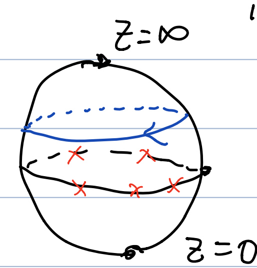
Let $w = 1/z$, then $\{|z| = 2\} \Leftrightarrow \{|w| = \frac{1}{2}\}$.

$$\left(-\oint_{\substack{\{|w|=\frac{1}{2}\} \\ \text{CCW}}}\right) \frac{1}{(\frac{1}{w})^5 - 1}\, d\left(\frac{1}{w}\right).$$

$$= -\oint_{|w|=\frac{1}{2}} \frac{w^5}{1 - w^5} \left(-\frac{1}{w^2}\right) dw.$$

$$= \oint_{|w|=\frac{1}{2}} \frac{w^3}{1 - w^5} \cdot dw,$$ no poles inside $|w| = \frac{1}{2}$.

$= 0$.



$\times$ = poles.

(c.f. HW #6. last question).

$C = \{|z|=1\}$.

(4). $\frac{1}{2\pi i}\int_{|z|=1} \frac{f'(z)}{f(z)}\,dz$ = # of zero of f inside C

− # of poles of f inside C.

$f = \frac{z^2(z+10)^3}{(z-10)^4}$ zero: $z=0$ order 2

$z=-10$ order 3

poles : $z=10$. order 4.

only $z=0$ is the zero inside C.

with order = 2

∴ by argument principle. we get 2.

$$\frac{f'(z)}{f(z)} = (\log f)'$$

$$\log\left(\frac{z^2(z+10)^3}{(z-10)^4}\right) = 2\log z + 3\log(z+10) - 4\log(z-10)$$

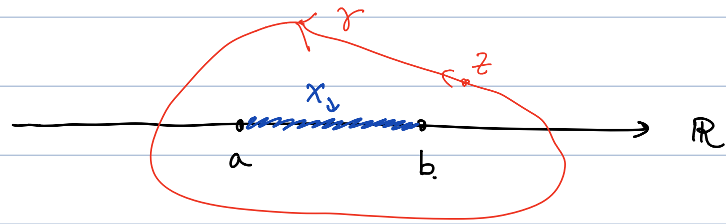
take derivative

$$\left[\log\left(\cdots\right)\right]' = \frac{2}{z} + \frac{3}{z+10} - \frac{4}{z-10}.$$

$$\frac{1}{2\pi i}\oint_{|z|=1} \left(- \cdots \right) dz \overset{C.I.F}{=\!=} 2.$$

#1

#3 : (1). $f(z) = \displaystyle\int_a^b \frac{g(x)}{z-x}\, dx.$

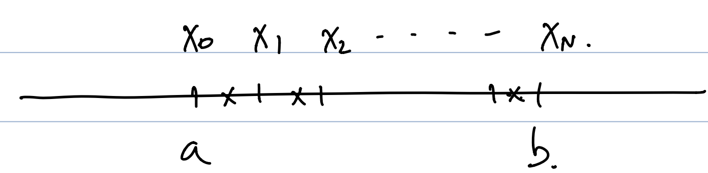


- $f(z)$ does not have isolated singularity
  the entire segment $[a,b]$ is the singular locus.

- $$\frac{1}{2\pi i}\oint_\gamma f(z)\, dz = \frac{1}{2\pi i}\int_{z\in\gamma}\left(\int_{x\in[a,b]} \frac{g(x)}{z-x}\, dx\right) dz$$

$$= \int_{x\in[a,b]}\left(\frac{1}{2\pi i}\int_\gamma \frac{g(x)}{z-x}\, dz\right) dx = \int_{x=a}^{b} g(x)\cdot dx.$$

$\because$ integrand is bounded
integration domain is compact.

- 

approximate $f(z) \approx \displaystyle\sum_{i=1}^{N} \left(\frac{g\left(\frac{x_i + x_{i-1}}{2}\right)}{z - \frac{x_i + x_{i-1}}{2}} \cdot \right) (x_i - x_{i-1})$

$$\simeq \int_{x=a}^{b} \frac{g(x)}{z-x}\, dx$$

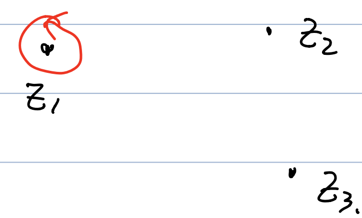
(2). by same argument $\oint_\gamma F(z)\, dz = 0$

(3) How to recover $g(x)$ by $f(z)$.

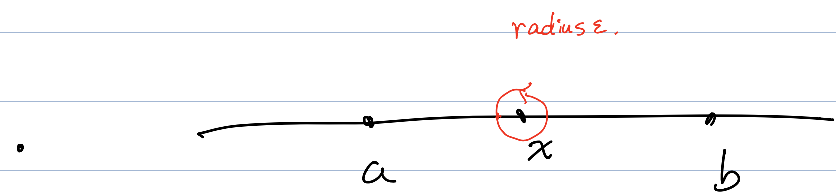
if we know

$$f(z) = \frac{a_1}{z-z_1} + \frac{a_2}{z-z_2} + \frac{a_3}{z-z_3}$$

~~if~~ how to extract a_i using ~~u~~ integration ?



$$a_1 = \frac{1}{2\pi i} \oint_{|z-z_1|=\varepsilon} f(z)\, dz.$$



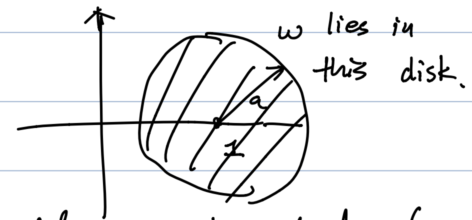
$g(x) = ?$

$$\frac{1}{2\pi i} \oint_{|z-x|=\varepsilon} f(z)\, dz = \int_{x-\varepsilon}^{x+\varepsilon} g(x')\, dx'.$$

$$g(x) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \cdot \int_{x-\varepsilon}^{x+\varepsilon} g(x')\, dx'$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \cdot \frac{1}{2\pi i} \cdot \oint_{|z-x|=\varepsilon} f(z)\, dz$$

4. (1). $w = 1 - az$ then
$|z| < 1$



thus $\log w$ is well-defines single valued funtion.

$$f(z) = \log(1-az), \quad f(0) = \log 1 = 0.$$

(2). $\log(1-az)$. $\overset{?}{\sim}$ $\log|1-a\cdot e^{i\theta}|$.

$$\mathrm{Re}(\log(1-az)) = \log|1-a\cdot e^{i\theta}|.$$

$$\int_0^{2\pi} \log|1-a\cdot e^{i\theta}| \cdot d\theta$$

$$= \int_0^{2\pi} \mathrm{Re}(\log(1-a\cdot e^{i\theta}))\cdot d\theta$$

$$= \mathrm{Re} \int_0^{2\pi} \log(1-a\cdot e^{i\theta})\cdot d\theta.$$

$\because$ $d\theta$ is real. let $z = e^{i\theta}$

$$= \mathrm{Re} \oint_{|z|=1} \log(1-az)\cdot \frac{dz}{iz}$$

$$= \mathrm{Re} \oint_{|z|} \frac{\log(1-az)}{iz}\, dz$$

$$\operatorname*{Res}_{z=0} \frac{\log(1-az)}{z} = 0.$$

$$= \operatorname{Re}(0) = 0.$$

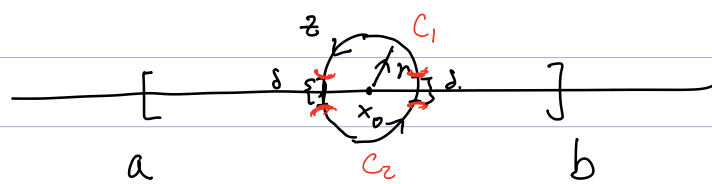
Alternatively : $u(z) = \operatorname{Re} \log(1-az) = \log|(1-az)|$ is a harmonic function

$\swarrow$ by Mean Value Thm.

$$\therefore \quad u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta})\, d\theta$$

$$\therefore \quad 0 = \log(1) = \frac{1}{2\pi} \int_0^{2\pi} \log|(1 - a\cdot e^{i\theta})|\, d\theta$$

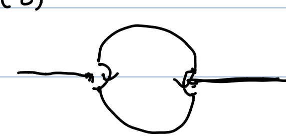
More about Problem 3. (#3).



$$\oint_{|z-x|=r} f(z)\, dz = \lim_{\delta \to 0} \left(\int_{C_1} f(z)\, dz + \int_{C_2} f(z) \cdot dz \right).$$

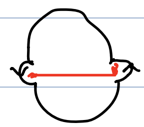
$$f(z) = \int_a^b \frac{g(x)}{z-x}\, dx = \underbrace{\int_{\substack{x \in [a,b] \\ x \notin [x_0 - r,\, x_0 + r]}} \frac{g(x)}{z-x} \cdot dx}_{f_2(z)} + \underbrace{\int_{x_0 - r}^{x_0 + r} \frac{g(x)}{z-x}\, dx}_{f_1(z)}.$$

Goal : $\oint f_2(z)\, dz = 0$:



$|z-x|=r$

$$\oint_{|z-x|=r} f_1(z)\,dz = \int_{x_0-r}^{x_0+r} g(x)\,dx$$



Ch5 Entire Functions.

- holomorphic. on the entire $\mathbb{C}$.

e.g. $\sin(z)$, e^z,

- if $|f(z)|$ bounded, $\Rightarrow$ $f(z) = \text{const}$

$\therefore$ the intersting functions, $\sup_{z\to\infty} |f(z)| = \infty$.

- Recurring thm: engineer some function with desired zero locations.

- if we want a function with finitely many zeros. $z_1, \dots, z_n$, then polynomial will do:

$$P(z) = (z-z_1)\cdots(z-z_n).$$

- How about functions with ∞ many zeros?

i.e. zero set $= \mathbb{Z}$?

$\underline{\sin(\pi \cdot z)}$ does the job.

- How about more general case ?
  can I have a function $f(z)$, s.t.

  $f(z) = 0$ if only if $z = n^2$?

  for some $n \in \mathbb{Z}$, $n \neq 0$.

- Try : $f(z) = (z-1^2)(z-2^2)(z-3^2)\cdots$

  infinitely many factors.

  $f(0) = 1^2 \cdot (-2^2)(-3^2)(-4^2) \cdots\cdots$ doesn't converge.