Today: • Follow Ahlfors. Ch 5.

• Conformal mapping : after thanksgiving.

Ch5: Infinite product and infinite series.

§1: Taylor series, Laurent series.

(Recall similar Thm in Stein) Thm 1: Suppose $f_n : \Omega_n \to \mathbb{C}$ hol'c function on open sets Ω_n. suppose $\Omega = \limsup_n \Omega_n$, i.e. any point in Ω is contain in Ω_n for sufficiently large n.

Suppose $f_n \to f$ uniformly on every compact subsets of Ω, then f is hol'c in Ω.

Ex: $f_n = \dfrac{z}{1+2z^n}$, $\Omega_n = \{ |z| < 2^{-\frac{1}{n}} \}$.

$\Omega = \{ |z| < 1 \}$.

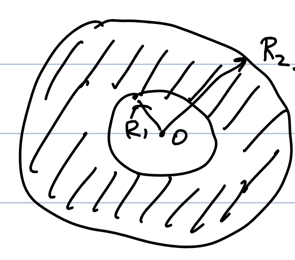
$\lim_{n\to\infty} f_n(z) = f(z) = z$ $\forall z \in \Omega$.

• Taylor Series: (recall).

If $f(z)$ is hol'c in Ω, if $z_0 \in \Omega$, then $\exists D_r(z_0)$, s.t. $\overline{D_r(z_0)} \subset \Omega$, and

$$f(z_0+u) = f(z_0) + f'(z_0)\cdot u + \frac{f''(z_0)}{2!}u^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}u^n + \cdots$$

if $|u| < r$.



• Laurent Series: (new).

Say $f(z)$ is hol'c in an annulus $R_1 < |z| < R_2$

then,

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n z^n}_{} + \sum_{n=-1}^{-\infty} a_n z^n.$$

valid for $z \in \{ R_1 < |z| < R_2 \}$.

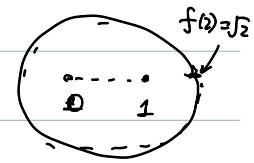
<u>Rmk</u>: • $f(z)$ doesn't necessarily have isolated singularity. e.g.

$$f(z) = \int_{-1}^{1} \frac{g(x)}{x-z}\, dx \qquad g(x): [-1,1] \to \mathbb{C} \text{ smooth.}$$

not isolated singularity

• or. $f(z) = \sqrt{z(z-1)}$

is hol'c in $\{ 2 < |z| < \infty \}$.



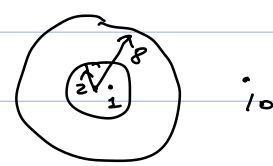
<u>Pf</u>: Find a decomposition:

$$f(z) = f_1(z) + f_2(z).$$

where $f_1(z)$ is hol'c in $\{ |z| < R_2 \}$.
$f_2(z)$ is hol'c in $\{ R_1 < |z| \}$.

e.g. $f(z) = \dfrac{1}{z+1} + \dfrac{1}{z+10}$

hol'c in $\{ 2 < |z| < 8 \}$



then let $f_1(z) = \underline{\dfrac{1}{z+10}}$, $f_2(z) = \dfrac{1}{z+1}$.

• <u>Construction</u>: for any $|z| < R_2$, choose r.
$|z| < r < R_2$.

$$f_1(z) = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(w)}{w-z} \cdot dw.$$

this doesn't dep on choice of r, as long as $|z|<r<R_2$

For $|z| > R_1$, let $|z| > r > R_1$,

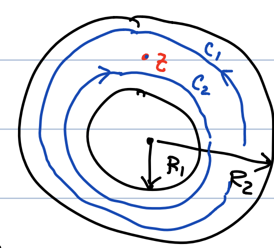
$$f_2(z) = \frac{-1}{2\pi i} \int_{|z|} \frac{f(w)}{w-z} \cdot dw.$$

Check : • $f_1(z)$ thus defined is hol'c for $|z|< R_2$

• $f_2(z)$ ———————— hol'c for $|z| > R_1$.

• for $R_1 < |z| < R_2$,

$$f_1(z) + f_2(z) = f(z).$$



find r_1, r_2, $R_1 < r_1 < |z| < r_2 < R_2$.

$$f(z) = \frac{1}{2\pi i} \int_{C_1 + C_2} \frac{f(w)}{w-z} \cdot dw.$$

$$= \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w-z} dw$$

$$= f_1(z) + f_2(z).$$

Use Taylor expansion.

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{valid for } |z| < R_2.$$

let $w = \frac{1}{z}$, $g(w) = f_2\left(\frac{1}{w}\right)$, $|z| > R_1 \iff |w| < \frac{1}{R_1}$

$g(w)$ is hol'c in $0 < |w| < \frac{1}{R_1}$, $g(0)=0$

$f_2(z) \to 0$, as $z \to \infty$. $\therefore$ $g(w) \to 0$, as $w \to 0$.

thus $w=0$ is a removable singularity. $\therefore$ $g(w)$ hol'c for $|w|<\frac{1}{R_1}$

$$g(w) = \sum_{n=1}^{\infty} b_n \cdot w^n \quad \text{valid for } |w| < \frac{1}{R_1}$$

$$\Rightarrow \quad f_2(z) = \sum_{n=1}^{\infty} b_n z^{-n} \quad \text{for any } |z| > R_1.$$

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot z^n + \sum_{n=1}^{\infty} b_n \cdot z^{-n}. \quad R_1 < |z| < R_2$$

Laurent expansion centered at $z=0$. #.

Ex: $f(z)=e^{\frac{1}{z}}$, hol'c for $0<|z|<\infty$.

$$f(z) = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots$$

- Laurent expansion of $\overset{f(z)}{\sin\left(\frac{1}{z-1}\right)}$ around $z=1$.

let $u = z-1$, then let $g(u) = f(1+u)$.

$$g(u) = \sin\left(\frac{1}{u}\right) = \frac{1}{u} + \frac{-1}{3!}\left(\frac{1}{u}\right)^3 + \cdots$$

(plugin Taylor expansion of $\sin x$, replace x by $\frac{1}{u}$.)

---

§2. Partial Fractions and Factorization.

Ex: Consider $f(z) = \frac{(z+1)(z+2)}{(z-2)(z-3)}$. a meromorphic function on $\mathbb{C}$,

- partial fraction presentation:

$$f(z) = \frac{(\ )}{z-2} + \frac{(\cdots)}{z-3} + \underline{(\ - - -\ )}$$

$\lim_{z\to 2} (z-2)\cdot f(z) = \frac{(2+1)(2+2)}{(2-3)}$

$\lim_{z\to\infty} f(z) = 1.$

- factorization presentation.

$$f(z) = \frac{(z+1)(z+2)}{(z-2)(z-3)}$$

specify the zero & poles.

Mittag-Leffler problem (for $\mathbb{C}$).

Q: Given a sequence of points $a_1, a_2, \cdots$ (pole location) such that $|a_i| \to \infty$ as $i \to \infty$; given non-constant polynomials $P_n(z)$, $n=1,2,\cdots$, can one find a meromorphic function $f(z)$ on $\mathbb{C}$, s.t.

$\forall n$, $\exists\, D_\varepsilon(a_n)$, $\forall z \in D_\varepsilon(a_n)$,

$$f(z) = P_n\left(\frac{1}{z-a_n}\right) + \text{regular part.}$$

Ex: $a_n = n$. $P_n(z) = z$.

we want a $f(z)$ have poles at $z=n$, $n=1,2,\cdots$

locally near $z=n$, $f(z) = \frac{1}{z-n} + (\text{regular})$

Try:

$$f(z) = \frac{1}{z-1} + \frac{1}{z-2} + \frac{1}{z-3} + \cdots\cdots$$

Does it make sense?

No. e.g. $f(0) = \frac{1}{-1} + \frac{1}{-2} + \frac{1}{-3} + \cdots\cdots$ divergent.

for any z, it is divergent.
($\because$ for n large $\frac{1}{z-n} \approx \frac{1}{-n}$.)

$$f_N(z) = \sum_{n=1}^{N} \frac{1}{z-n}$$

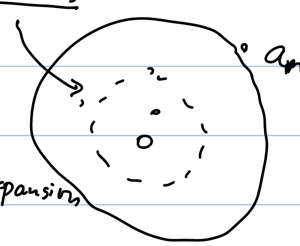
for any N finite. $f_N(z)$ behaves nicely, have designed poles and singular terms near the poles.

Approach: $$f(z) = \sum_{n=1}^{\infty} \left(P_n\left(\frac{1}{z-a_n}\right) - \underbrace{q_n(z)}_{\text{polynomial}} \right)$$

· For simplicity, assume $a_n \neq 0$, then.

we can let $q_n(z)$ be the "first few terms" of the Taylor expansion of $P_n\left(\frac{1}{z-a_n}\right)$ at $z=0$.

$q_n(z)$ = sum of first M_n terms in $P_n\left(\frac{1}{z-a_n}\right)$ Taylor expansion



By choosing M_n large enough, we can make.

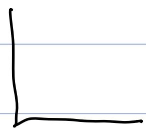
$$P_n\left(\frac{1}{z-a_n}\right) - q_n(z) \quad \text{small inside } D_{\frac{1}{2}|a_n|}(0)$$

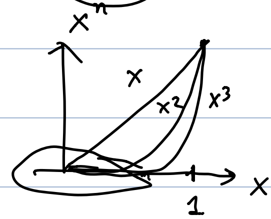
$$\frac{1}{z-1} = -\frac{1}{1-z} = -(1+z+z^2+z^3+\cdots) \quad \text{for } |z|<1.$$

for example.

$$q(z) = -(1+z+z^2+z^3).$$

$$\frac{1}{z-1} - q(z) = -(z^4 + z^5 + \cdots).$$





the higher power x^n has the smaller $|x^n|$ is near $x=0$.

We can choose m_n to be large enouch, such thats

$$\left| P_n\left(\frac{1}{z-a_n}\right) - q_n(z) \right| < \frac{1}{n^2} \quad \forall\ |z| \leq \frac{1}{2}|a_n|.$$

Finally, check for any $z \in \mathbb{C}$.

$z \notin \{a_1, a_2, \cdots\}$.

$f(z)$ is finite. We consider those a_n, s.t. $|a_n| > 2|z|$, there is only finitely many a_n, with $|a_n| < |z|$.

$$\left| \sum_{\substack{n=1,2,\cdots \\ |a_n|>2|z|}} P_n\left(\frac{1}{z-a_n}\right) - q_n(z) \right|$$

$$\leq \sum_{\substack{n=1,2,\cdots \\ |a_n|>2|z|}} \left| P_n\left(\frac{1}{z-a_n}\right) - q_n(z) \right|.$$

$$\leq \sum_{\substack{n=1,2,\cdots \\ |a_n|>2|z|}} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

$$f(z) = \sum_{|a_n| \le 2|z|} \left(P_n\left(\frac{1}{z-a_n}\right) - q_n(z)\right)$$

finitely many terms

$$+ \sum_{|a_n| > 2|z|} \left(P_n\left(\frac{1}{z-a_n}\right) - q_n(z)\right).$$

bounded.

Back to the example. Need to find $q_n(z)$. s.t.

$$\left| \frac{1}{z-n} - q_n(z) \right| < \frac{1}{n^2} \qquad \text{for } |z| < \frac{n}{2}.$$

$|z|<n$.

$$\frac{1}{z-n} = -\frac{1}{n}\,\frac{1}{1-\frac{z}{n}}$$

$$= -\frac{1}{n}\left(1 + \frac{z}{n} + \left(\frac{z}{n}\right)^2 + \cdots\cdots\right)\cdots$$

$\because \left|\frac{z}{n}\right| < \frac{1}{2}$, need M_n, s.t. $\frac{1}{n}\left(\frac{1}{2}\right)^{M_n} < \frac{1}{2n^2}$.

e.g. $M_n = \frac{\log n}{\log 2} \cdot 10$.

$$\frac{1}{z-10}$$