Today: Follow Ahlfors Ch5.

- example of infinite sum (with prescribed poles)
- infinite product (with prescribed zeros).

Recall last time:

Want build ~~an~~ a meromorphic function on $\mathbb{C}$ $f(z)$. such that $f(z)$ has poles at $a_1, a_2, \dots$, $|a_n| \to \infty$ as $n \to \infty$.

and near each pole $a_n$, we want

$$f(z) = P_n\left(\frac{1}{z-a_n}\right) + \text{regular part.} \qquad z \sim a_n.$$

$P_n(\cdot\cdot)$ polynomial., $\deg \geq 1$.

Construction:
$$f(z) = \sum_{n=1}^{\infty} \left( P_n\left(\frac{1}{z-a_n}\right) - q_n(z) \right)$$

$q_n(z)$ is the first few terms of the Taylor expansion of $P_n\left(\frac{1}{z-a_n}\right)$ near $z=0$.

Intuition: by adding this correction term $-q_n(z)$, we minimize the "pollution" to the "constructed region", i.e. $\{|z|<R\}$.

Example: (1) $\frac{\pi^2}{\sin^2 \pi z}$. it has pole at $z \in \mathbb{Z}$.

near $z=n$, $\quad \frac{\pi^2}{\sin^2 \pi z} = \frac{1}{(z-n)^2} + \text{reg. part.}$

✓

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{+\infty} \frac{1}{(z-n)^2} + g(z)$$

↑ entire function.

↑ Does this make sense?

- this infinite sum is convergent. $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

Claim:

- $g(z) = 0$.

① $\frac{\pi^2}{\sin^2(\pi z)}$ is periodic under $z \to z+m$, $m \in \mathbb{Z}$.

so is $\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} \Rightarrow g(z) = g(z+1)$.

② ✓ $\frac{\pi^2}{\sin^2(\pi z)} \to 0$ (uniformly in x) as $z = x+iy$, $|y| \to \infty$.

also ✓. $\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} \to 0$, as $|y| \to \infty$.

$\sin(a+b) = \sin a \cos b + \sin b \cdot \cos a$.

$$|\sin^2(\pi(x+iy))| = |\sin(\pi x)\cdot \cos(i\pi y) + \sin(i\pi y)\cos(\pi x)|^2$$

$a \in \mathbb{R}$.

- $\cos(i\cdot a) = \frac{e^a + e^{-a}}{2} = \cosh(a)$

$\sin(ia) = \frac{e^{i(ia)} - e^{-i(ia)}}{2i} = i\,\frac{e^a - e^{-a}}{2} = i\cdot\sinh(a)$

$$= |\sin\pi x \cdot \cosh\pi y + i\cdot \sinh \pi y \cdot \cos\pi x|^2$$

$(\cos a)^2 + (\sin a)^2 = 1$

$(\cosh a)^2 - (\sinh a)^2 = 1$

$$= (\sin\pi x)^2\cdot(\cosh\pi y)^2 + (\sinh\pi y)^2\cdot(\cos\pi x)^2.$$

$$= (1-\cos^2\pi x)(\cosh^2\pi y) + (\cosh^2\pi y - 1)\cos^2\pi x.$$

$$= \cosh^2\pi y - \cos^2\pi x. \longrightarrow \infty \quad \text{as } |y| \to \infty$$

$\Rightarrow \quad g(z) \to 0$ uniformly in $x$, as $|y| \to \infty$.

$\Rightarrow \quad g(z)$ is bounded for $x \in [0,1]$, $y \in \mathbb{R}$.

and by periodicity, $g(z)$ is bounded in $\mathbb{C}$.

$\Rightarrow \quad g(z) = \text{const}$ by Liouville. thm.

and as $\lim_{y \to \infty} g(x+iy) = 0$, $g(z) = 0$.

Prop:

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$$

Read in Ahlfors P190. about expansion of

- $\pi \cdot \cot(\pi z) = \pi \cdot \frac{\cos(\pi z)}{\sin(\pi z)}$.
- $\frac{\pi}{\sin(\pi z)}$.

Infinite Product.

- Let $b_1, b_2, b_3 \cdots$ be non-zero complex numbers.

we say $\prod_{i=1}^{\infty} b_i$ converge, if the partial products

$$\Pi_N = \prod_{i=1}^{N} b_i, \quad \text{converges as a sequence.}$$

Then we say, the infinite product is.

$$\Pi = \lim_{N \to \infty} \Pi_N.$$

- We are interested in the case, where $\Pi$ exists. and is non-zero.

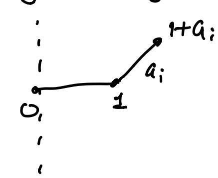
Consider the case $b_i = 1 + \underline{a_i}$, and $a_i \to 0$.

<u>Lemma</u>: $\prod_{i=1}^{\infty} b_i$ exist and is non zero.

if. $\sum_{n=1}^{\infty} |a_i| < \infty$

<u>Pf</u>: · w.l.o.g. we may assume all $a_i$, satisfies $|a_i| < \frac{1}{2}$.

· Then we may define $\log(1+a_i)$, by specify $\arg(1+a_i) \in (-\pi, \pi)$.




· $\sum_{i=1}^{\infty} \log(1+a_i)$ exist

$\Leftrightarrow$ $\prod_{i=1}^{\infty} b_i$ exists and is non zero.

· $\log(1+a_i) \approx a_i$ for $a_i$ small.

more precisely, there exists $\varepsilon > 0$, and $N > 0$, s.t. $\forall$ $i > N$,

$$(1-\varepsilon)|a_i| < \left|\log(1+a_i)\right| < (1+\varepsilon)|a_i|$$

hence if $\sum |a_i| < \infty$, then $\sum \left|\log(1+a_i)\right| < \infty$.

$\Rightarrow \sum \log(1+a_i) < \infty \Rightarrow \prod_{i=1}^{\infty} b_i$ finite.

· Consider an entire function $f(z)$. (on $\mathbb{C}$). with $m$-order of zero at $z=0$, and others roots at $a_1, a_2, a_3, \cdots$ (possibily with repetitions). $|a_i| \to \infty$ as $i \to \infty$.

Naive guess: $f(z)$ can be written as.

$$f(z) = z^m \cdot \underline{\prod_{i=1}^{\infty}\left(1-\frac{z}{a_i}\right)} \cdot \underbrace{e^{g(z)}}_{\text{a. non-vanishing entire function.}}$$

- it works if $\sum_{i=1}^{\infty} \frac{1}{|a_i|} < \infty$,

since in this case. $\prod_{i=1}^{\infty}\left(1-\frac{z}{a_i}\right)$ converge.

- Cure: need to change the naive factor $\left(1-\frac{z}{a}\right)$ to a "canonical factor"

$$\underline{E_k(z;a)} = \underbrace{\left(1-\frac{z}{a}\right)\cdot e^{\frac{z}{a}+\frac{1}{2}\left(\frac{z}{a}\right)^2+\cdots+\frac{1}{k}\left(\frac{z}{a}\right)^k}}$$

If $\left|\frac{z}{a}\right| < 1$, then we have.

working over the principal branch. of log.

$$\log E_k(z;a) = \underline{\log\left(1-\frac{z}{a}\right)} + \underbrace{\frac{z}{a}+\frac{1}{2}\left(\frac{z}{a}\right)^2+\cdots+\frac{1}{k}\left(\frac{z}{a}\right)^k}_{\text{the first few terms of Taylor expansion of } -\log\left(1-\frac{z}{a}\right).}$$

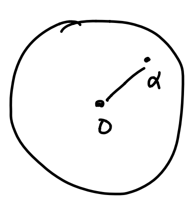
$|\alpha| < 1$,

$$\frac{1}{1-\alpha} = 1+\alpha+\alpha^2+\cdots$$

integrate $\alpha$.

$$\int_0^{\alpha} \frac{1}{1-w}\cdot dw.$$

$$= -\log(1-\alpha).$$



$$-\log(1-\alpha) = \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{3} + \cdots$$

If we replace $\left(1-\frac{z}{a_n}\right)$ by $E_{k_n}(z, a_n)$, and choose $\underline{k_n}$ large enough for each $n$, then $\forall R>0$,

$$\prod_{n=1}^{\infty} E_{k_n}(z, a_n) \text{ is uniformly convergent}$$

on this closed disk. $\{|z| \le R\}$. (equivalently, for any compact subset $K \subset \mathbb{C}$)

(Read proof in Ahlfors Ch 5. §2.2).

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(Ahlfors)

Ch5. §3 Jensen's Formula.

- Recall, if $f(z)$ is hol'c in $\Omega$, $\Omega \supset \overline{\mathbb{D}}$.

then

- $$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, d\theta.$$

- $u(z) = \operatorname{Re}(f(z))$. $u(z)$ harmonic.

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, d\theta.$$

- If $f(z)$ is hol'c and $\boxed{\text{non-vanishing}}$ on $\overline{\mathbb{D}}$. then $\underline{\log|f(z)|}$ is a well-defined, ~~note~~ harmonic function.

( harmonic, $\because$ $\mathrm{Re}\cdot \log(f(z)) = \log|f(z)|$ ).

choose whatever branch of log. since the ambiguity is $2\pi i \cdot \mathbb{Z}$.

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\cdot d\theta. \quad (*)$$

- What if $f$ has a zero on the boundary $\partial\mathbb{D}$? (but $f(z)\neq 0$. $\forall z\in\mathbb{D}$).

$z_0$ on $\partial\mathbb{D}$, $|z_0|=1$.

$$f(z) = \underline{F(z)}\cdot(z-z_0). \qquad F(z) \text{ is non-vanishing on } \overline{\mathbb{D}},$$

then

$$\log|F(0)| = \frac{1}{2\pi}\cdot\int_0^{2\pi} \log|F(e^{i\theta})|\cdot d\theta.$$

$|z_0| = 1$.

$$\log|F(0)| = \log\left|\frac{f(0)}{0-z_0}\right| = \log|f(0)|.$$

$$\log\left|\frac{f(e^{i\theta})}{e^{i\theta}-z_0}\right| = \log|f(e^{i\theta})| - \log|e^{i\theta}-z_0|$$

Claim: $$\int_0^{2\pi} \log|e^{i\theta}-z_0|\cdot d\theta = 0.$$

$\because z_0 = e^{i\varphi}$,

$$\int_0^{2\pi}\log|e^{i(\theta-\varphi)}-1|\,d\theta = 0 \quad \text{(HW problem)}$$

$$\therefore \int_0^{2\pi}\log|F(e^{i\theta})|\,d\theta = \int_0^{2\pi}\log|f(e^{i\theta})|\,d\theta.$$

$\therefore$ If $f(z)$ has zeros on the boundary $\partial \mathbb{D}$.

then. $$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})| \cdot d\theta. \quad \left(\substack{\text{same} \\ *}\right)$$

- What if $f(z)$ has zeros $a_1, \cdots, a_n$ inside $\mathbb{D}$?

but $a_i \neq 0$.

then

(Jensen formula) $$\log|f(0)| = \underbrace{\sum_{i=1}^n \log|a_i|}_{\text{extra factors.}} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})| d\theta.$$

$$\underline{f(z) = F(z)\,(z-a_1)\cdots(z-a_n)}.$$

Then $F(z)$ is non-vanishing for $z \in \overline{\mathbb{D}}$.

By (*): $\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(e^{i\theta})| \cdot d\theta.$

$$\frac{1}{2\pi}\int_0^{2\pi} \log|F(e^{i\theta})| \cdot d\theta.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\log|f(e^{i\theta})| - \sum_{i=1}^n \log|e^{i\theta} - a_i|\right) d\theta.$$

$|a_i| < 1$.

$$\frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - a_i| \cdot d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \log|1 - a_i \cdot e^{-i\theta}| \cdot d\theta = 0.$$

$$= \frac{1}{2\pi} \cdot \int_0^{2\pi} \log|f(e^{i\theta})| \cdot d\theta.$$

$$\underline{\log|F(0)|} = \log|f(0)| - \sum_{i=1}^n \log|a_i|$$

$$\log|f(0)| - \sum_{i=1}^n \log|a_i| = \frac{1}{2\pi} \cdot \int_0^{2\pi} \log|f(e^{i\theta})| d\theta$$

move to the right.

$$\log|f(0)| = \sum_{i=1}^{n} \log|a_i| + \frac{1}{2\pi} \cdot \int_0^{2\pi} \log|f(e^{i\theta})|\, d\theta.$$