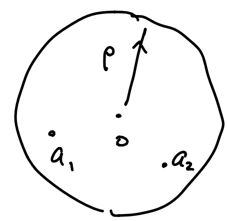
Today: Ahlfors Ch 5. §3, §5.

§3. Jensen formula:

- Say f is hol'c on $\overline{\mathbb{D}_\rho(0)}$, and assume $f(0) \neq 0$.
  and $a_1, a_2, \dots, a_n \in \mathbb{D}_\rho(0)$ are the zeros of f. (repeated possibly)

$$\log|f(0)| = -\sum_{i=1}^{n} \log\left|\frac{\rho}{a_i}\right| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\theta})|\,d\theta.$$

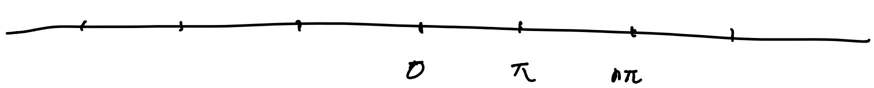
(last time. $\rho = 1$. $\log|a| = -\log\left|\frac{1}{a}\right|$)



- Hadamard's Thm:
  - relates the number of zeros with the growth rate of a function.

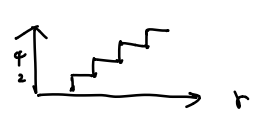
- Ex: $f(z) = \sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$.

  - $f(z)$ has roots at $n\pi$, $n \in \mathbb{Z}$.
  - $|f(z)| \leq \dfrac{|e^{iz}| + |e^{-iz}|}{2} \leq e^{|z|}$.



- $\nu(r) :=$ # of roots of $f(z)$ inside $\{|z| < r\}$.

(then for $f = \sin(z)$,



, $\nu(r) \leq a \cdot r$ linear in r)

- $M(r) := \sup_{|z|\le r} |f(z)| = \sup_{|z|=r} |f(z)|$.

(then. for $f(z) = \sin(z)$. $M(r) \le e^{r}$ linear in r.)

<u>Concepts</u> :
- genus h.
- order (of growth)

- Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.
the genus h of f is the smallest non-negative integer, such that. we have

$$f(z) = e^{g(z)} \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \cdot e^{\overbrace{\left(\frac{z}{a_n}\right) + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h}^{h \text{ terms}}}$$

and $g(z)$ is a polynomial of degree $\le h$.

<u>Ex</u> :
- $f(z) = e^{z^2}$, $h = 2$.
- $f(z) = \sin(z)$. , $h = 1$.

[related to Hw. #3. $\prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) \cdot e^{z/n}$ is convergent.]

Def: order λ of a function f.

$$\lambda = \inf \{ \lambda' \mid \underline{M(r)} \le e^{r^{\lambda'}} \}.$$ real number

- Ex: $f(z) = e^{z^2}$. $\lambda = 2$.

Thm (Hadamard): If f is an entire function, then

$$h \le \lambda \le h+1.$$

remark: if λ is not an integer, then h is uniquely determined.

§5. Normal Family.

A family of hol'c functions over an open set Ω, (with some common properties), and we are looking for limits for a sequence of functions.

- Let's consider functions from Ω ($\subset \mathbb{C}$ open subset), valued in a "metric space" S.

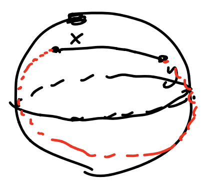
- Recall a metric space S: is a set with a distance function: $d(x,y) \in \mathbb{R}_{\ge 0}$. $x, y \in S$.
  - $d(x,y) = 0 \iff x = y$

- $d(x,y) + d(y,z) \geq d(x,z)$
- $d(x,y) = d(y,x)$.

Ex: • Euclidean metric on $\mathbb{R}^n$

$$d(x,y) = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}, \quad x, y \in \mathbb{R}^n$$

• Sphere, $S^2 \subseteq \mathbb{R}^3$ unit sphere.



- find a "tight line" = geodesic between x and y, on the sphere.
- measure its length

here: the geodesic connecting x and y will be a segment of a great circle passing through x, y.

• $\mathbb{C}$ has 2 metrics.

- $\mathbb{C} \simeq \mathbb{R}^2$, Euclidean metric.
- $\mathbb{C} \subset \hat{\mathbb{C}} \simeq S^2$; use the sphere metric.

$\mathbb{R}^2 \simeq \mathbb{C}$ metric space. general metric space.

• A function: $f : \Omega \to (S, d)$ continuous

$Map(\Omega, S)$. • this is a set.

• can we put a metric on this

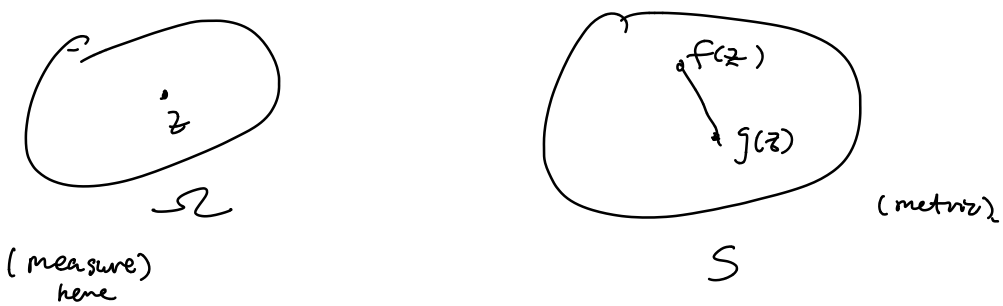
set ?

Several different ways.

$1 < p < \infty$

• L^p- distance. $f, g \in Map(\Omega, S)$

$$\rho(f, g) = \left| \int_{\Omega} \left[d\left(f(z), g(z)\right) \right]^p \, d\,Vol(z) \right|^{\frac{1}{p}}$$



• L^∞ - distance. (sup norm).

$$\rho(f, g) = \sup_{z \in \Omega} d\left(f(z), g(z)\right).$$

if we use L^∞- distance as metric on $Map(\Omega, S)$.
then $f_n \xrightarrow{L^\infty} f$ converges., it means
$f_n \to f$ uniformly on Ω.

• If we want to describe " $f_n \to f$
on every compact subset $E \subset \Omega$, how to
construct a metric ?

- Construction of such metric in 3 steps:

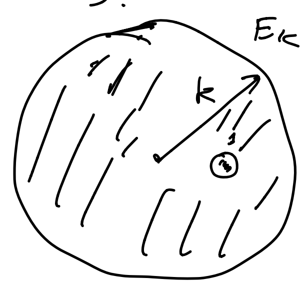
1). Find an exhausting seq of compact subsets

$$E_1 \subset E_2 \subset E_3 \subset \cdots, \quad E_i \subset \Omega, \quad \Omega = \bigcup_{i=1}^{\infty} E_i$$

- For example, construct E_k, s.t.

$$E_k = \{ z \in \Omega \mid \underline{|z| \le k}, \ \underline{d(z, \partial\Omega) \geq \tfrac{1}{k}} \}.$$

ex: - if $\Omega = \mathbb{C}$. $\quad E_k = \{ |z| \le k \}$.



- if $\Omega = \mathbb{C} \setminus \{1\}$.

E_k is growing in radius, and also approach $\partial\Omega$.

2). Modify the metric on S so that S has bounded diameter. $\quad \alpha \mapsto \frac{\alpha}{1+\alpha}$

$\forall\ x, y \in S$

$$\delta(x,y) = \frac{d(x,y)}{1+d(x,y)} \le 1$$

one can check. (S, δ) also satisfies the distance function conditions.

3). For any $f, g \in \underline{\mathrm{Map}(\Omega, S)}$,

(not distance functions) $\cdot\ \underline{\rho_{E_k}}(f,g) = \sup_{z \in E_k} \delta(f(z), g(z)) \le 1$

$$\cdot \ \boxed{\rho(f,g).} = \sum_{k=1}^{\infty} \underbrace{\rho_{E_k}(f,g)}\cdot 2^{-k}.$$
$$\leq \sum_{k=1}^{\infty} \cdot 2^{-k} = 1.$$

$\left(Map(\Omega, S), \quad \rho \right)$ is a metric space.

<u>check</u> : $\cdot$ ρ is a distance function.

(1) $\rho(f,g) = 0 \iff f = g.$? ✓.

$\because$ $\rho(f,g) = 0 \Rightarrow \rho_{E_k}(f,g) \ \forall k$

$\Rightarrow f = g$ on E_k $\forall k.$

$\Rightarrow \because \Omega = \bigcup_{k=1}^{\infty} E_k$, $\therefore f = g$ on Ω.

$$\left.\begin{array}{ll} (2) & \rho(f,g) = \rho(g,f) \\ (3) & \rho(f,g) + \rho(g,h) \geq \rho(f,h) \end{array}\right\} \because \rho_{E_k}(\cdot,\cdot)$$
satisfies these ~~properties~~ properties.

<u>check</u>: If $\{f_n\}$ is a seq of fcn in $Map(\Omega, S)$, then $\left(f_n \to f \text{ w.r.t. the } \rho \text{ distance} \right)$

$\iff \left(f_n \to f \right.$ uniformly for every compact subset $\left. E \subset \Omega \right)$.

Pf: $\Rightarrow$. since $E \subset \Omega$, $\Omega = \bigcup E_k$.

$$E = \bigcup_{k=1}^{\infty} (E \cap E_k).$$

$\because E$ is compact $\therefore$ $E = E \cap E_K$ for K. large i.e. $E \subset E_K$ for some large enough K. (Ex : why?) enough.

want show $\rho_E(f_n, f) \to 0$.

but $E \subset E_K$, $\rho_E(f_n, f) < \rho_{E_K}(f_n, f)$.

$$\rho(f_n, f) = \frac{1}{2}\rho_{E_1}(f_n, f) + \frac{1}{2^2}\cdot\rho_{E_2}(f_n, f) + \cdots + \underbrace{\frac{1}{2^k}\rho_{E_K}(f_n, f)}_{} + \cdots \geq \frac{1}{2^k}\rho_{E_K}(\cdots)$$

$$\rho_{E_K}(f_n, f_n) \leq 2^K \cdot \rho(f_n, f).$$

but K is fixed., and $\rho(f_n, f) \to 0$. by assumption

$$\therefore \quad \rho_E(f_n, f) \leq \rho_{E_K}(f_n, f) \leq 2^k \cdot \rho(f_n, f) \to 0$$

as $n \to \infty$.

$\Leftarrow$: we have uniform convergence for every E_K. We need to show, $\forall \varepsilon, \exists N$, s.t. $\forall n > N$,

$$\rho(f_n, f) < \varepsilon.$$

Find k_0 large enough, s.t. $2^{-k_0} < \frac{\varepsilon}{2}$.

then

$$\rho(f_n, f) \leq \sum_{k=1}^{k_0} 2^{-k}\cdot\rho_{E_K}(f_n, f) + \underbrace{\sum_{k=k_0+1}^{\infty} 2^{-k}}_{= 2^{-k_0}}$$

$$\leq \sum_{k=1}^{k_0} 2^{-k}\cdot\rho_{E_K}(f_n, f) + \frac{\varepsilon}{2}.$$

Now, pick $\underline{\underline{N}}$ large enough, s.t. $\forall 1 \le k \le k_0$,

$$\rho_{E_k}(f_n, f) < \frac{\varepsilon}{2} \qquad \forall n > N$$

for $n > N$. $\le \sum_{k=1}^{k_0} 2^{-k} \cdot \left(\frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2}$

$$\le \frac{\varepsilon}{2} \cdot 1 + \frac{\varepsilon}{2} = \varepsilon.$$