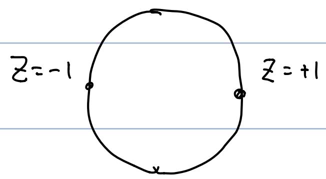
Today: ① multivalued function

② Fourier series & harmonic functions.

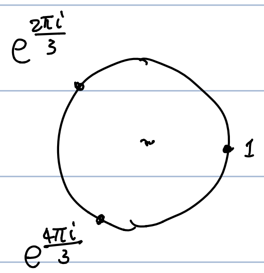
① Multi valued functions:

motivation: ① $z^2 = 1$ $z = ?$ ± 1



square roots of unity.

$z^3 = 1$, $z^3 = e^{2\pi i} = e^{4\pi i} = e^{6\pi i} \cdots, 1,$



$z = e^{\frac{2\pi i}{3}}, \; e^{\frac{4\pi i}{3}}, \; 1.$

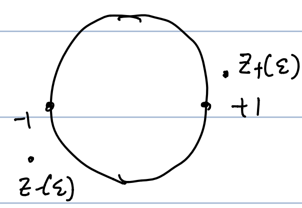
cubic roots of unity.

$z^n = 1$, $z = e^{\frac{2\pi i}{n} \cdot j}$ $\underline{j} = 0, 1, \cdots, n-1.$

n many roots.

② how about change 1 to something else?

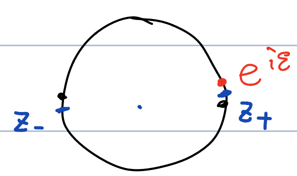
$z^2 = 1 + \varepsilon.$



expectation: get 2 roots. z_+, z_-.

near the original roots ± 1,

- For example, $z^2 = \underline{e^{i\varepsilon}}$ $e^{i\varepsilon}$



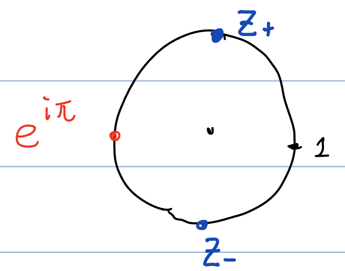
$z_+ = e^{i\varepsilon/2}$, $z^2 = e^{i\varepsilon} = e^{2\pi i + i\varepsilon}$

$z_- = e^{\pi i + i\frac{\varepsilon}{2}}$

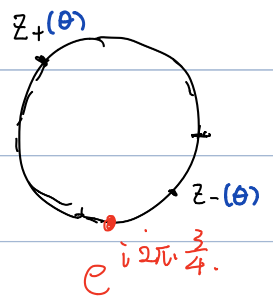
• Now. consider $z^2 = e^{i\theta}$ $\quad \theta \in [0,\pi]$

$\Rightarrow \quad z_+ = e^{i\theta/2}, \quad z_- = e^{\pi i + i\frac{\theta}{2}} = -e^{i\frac{\theta}{2}}.$

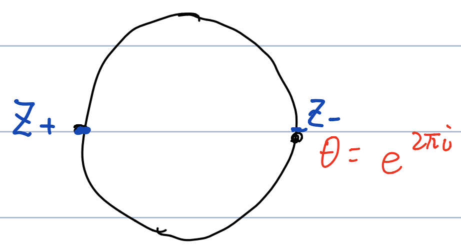
as $\theta = \pi$. $\quad z_+ = i, \quad z_- = i$.



• let θ ranges from 0 to 2π.



$\rightsquigarrow$



$z^2 = e^{i\theta}$. $\quad \theta = 0 \rightsquigarrow \theta = 2\pi$.

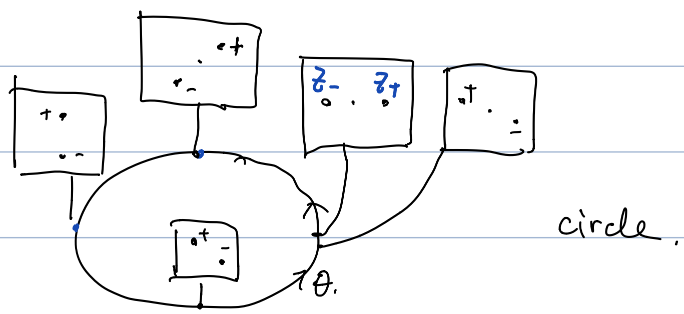
• as $e^{i\theta}$ goes to $e^{2\pi i}$, or return to 1.

z_+ and z_- switch positions.

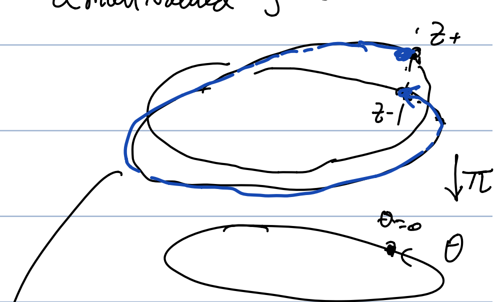
$z_+(\theta = 2\pi) = z_-(\theta=0) = -1$

$z_-(\theta = 2\pi) = z_+(\theta = 0) = 1$.

• Now imagine a function. input $= \theta$, output $= \pm\sqrt{e^{i\theta}}$



graph of a multivalued function.



boundary of a Möbiüs strip.

- Ex: do the same for $z^3 = e^{i\theta}$.

Remark: (1) this is an example of covering space, $\left(S^1 = \text{1-dim sphere.}\right)$

- for $z^2 = e^{i\theta}$, $S^1 \xrightarrow{2:1} S^1$
- for $z^n = e^{i\theta}$, $S^1 \xrightarrow{n:1} S^1$

(2) monodromy of the base loop. acts on the fiber (= preimage of the π)., by permute the points in the fiber.

let θ_0 be a point in the base S^1, say $\theta_0 = 0$.

let γ be a loop starting and ending on θ_0,

let F_0 be the fiber above θ_0, i.e. solutions of $z^2 = e^{i\theta}$, then, as θ moves along γ, and return to θ_0, we have a map : $T: F_0 \to F_0$,

sending. $\alpha \longmapsto \alpha'$ is the image in F_0 ~~are~~ after we keep track of α along the loop γ.

- Consider equation: $z^2 = c$, $c \in \mathbb{C}$

$c = r \cdot e^{i\theta}$, then $z = \sqrt{r}\, e^{i\theta/2}$, $\sqrt{r} \cdot e^{i\theta/2 + i\pi}$.

over each $c \neq 0$, we have 2 roots,

over $c = 0$, $z = 0$, only 1 root. (but with multiplicity 2)

$z = \sqrt{c}$.

$$\mathbb{C} \xrightarrow{2:1} \mathbb{C} \quad \text{cover}$$

$$z \longmapsto z^2.$$

$f(c) = \sqrt{c}$ is a multivalued function,

input $= c$

output $= \{z \mid z^2 = c\}$.

* **Logarithm**:

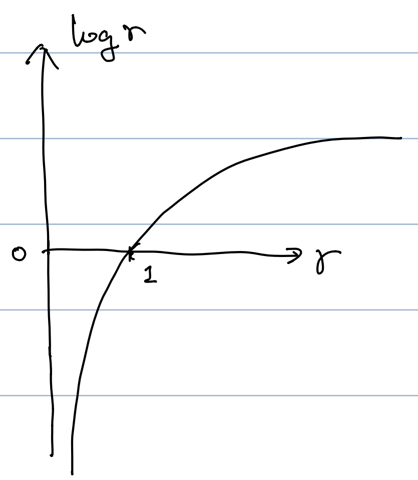
$$\log(z) = \{w \mid e^w = z\}.$$

input : $z \in \mathbb{C} \setminus \{0\}$.

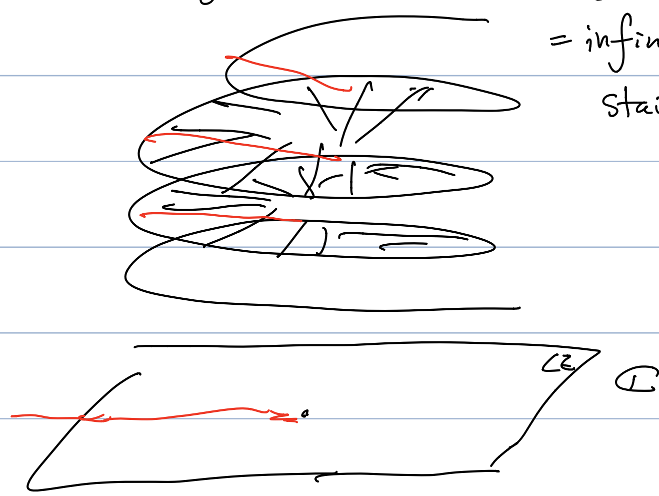
ouput : solution to equation $e^w = z$.

if $z = r \cdot e^{i\theta}$, $\log z = \log r + i \cdot \theta + 2\pi i \cdot \mathbb{Z}$.

↑ a single value.

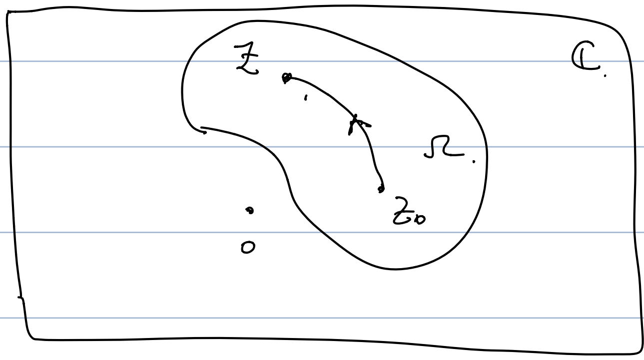


graph of $\log z$ = infinite staircase.



* Branch, Branch cut. (only for log).

- if we have a <u>simply connected</u> ~~domain~~ region. $\Omega \not\ni 0$



- and consider the function $f(z)$, satifying
  - $\begin{cases} f'(z) = \frac{1}{z}. \\ f(z_0) = c_0. \end{cases}$ for some $z_0 \in \Omega$.

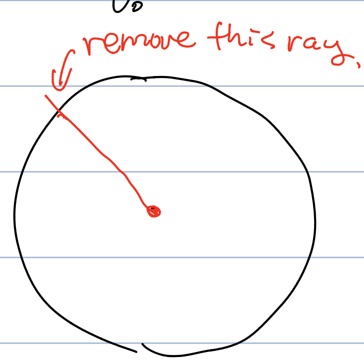
$\Rightarrow$ $\underline{f(z)} = \log(z) + \text{const.}$ $\qquad \log(z/z_0) + c_0.$

and $f(z)$ is single valued.

Indeed.
$$f(z) = c_0 + \int_{z_0}^{z} \frac{1}{z}\, dz.$$
$$=: c_0 + \log_{\Omega}(z/z_0).$$

<u>so</u>: if we choose $\Omega_{\theta_0} = \mathbb{C} \setminus \{r \cdot e^{i\theta_0} : r \in \mathbb{R}_{\geq 0}\}$

θ_0 is fixed.

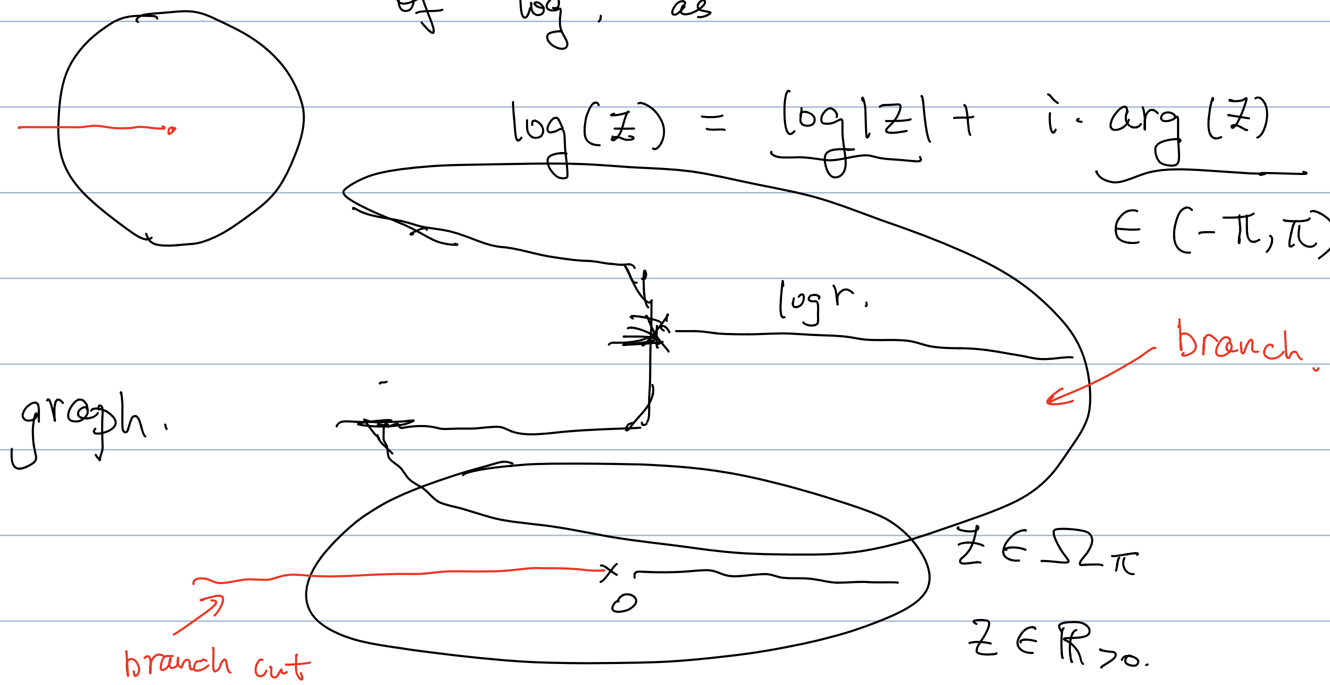


Ω_{θ_0} is simply connected.

on Ω_π ; , we define the principal branch of log, as

$$\log(z) = \underline{\log|z|} + i \cdot \underline{\arg(z)}$$

$\in (-\pi, \pi)$.



- a branch is a <u>choice</u> of the one of the output set for a multivalued function, that choice varies continuous as the input varies outside of the <u>branch cut</u>

- How to "cook up" a function on the unit disk from a function on the unit circle.

(1) if we have $f : \overline{\mathbb{D}} \to \mathbb{C}$ a hol'c function. then for any $z \in \mathbb{D}$

$$f(z) = \frac{1}{2\pi i} \oint_{|w|=1} \frac{f(w)}{w-z} \, dw$$

in HW.

(2) if we have $u : \overline{\mathbb{D}} \to \mathbb{R}$ a harmonic function, (e.g. $u = \mathrm{Re}(f)$, f : hol'c), then

Poisson integral formula

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) \cdot u(e^{i\theta}) \cdot d\theta.$$

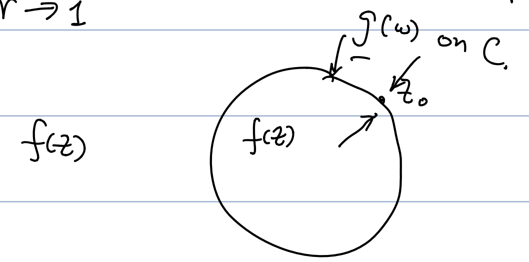
unit circle

(3) If $g(w) : C \to \mathbb{C}$, is an arbitrary <u>smooth</u> function, then

$$f(z) = \frac{1}{2\pi i} \oint_{|w|=1} \frac{g(w)}{w-z} \, dw. \qquad |z| \neq 1.$$

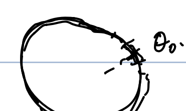
for $|z_0|=1$,

$$\lim_{r\to 1^-} f(rz_0) - \lim_{r\to 1^+} f(r \cdot z_0) = g(z_0).$$



(4) Let $U : C \to \mathbb{R}$ be any smooth function,

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) \cdot U(\theta) \cdot d\theta$$



then

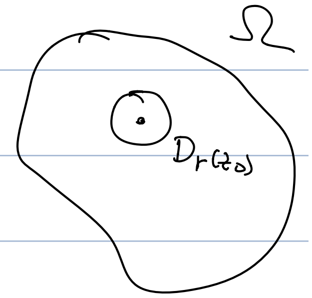
$$\lim_{r\to 1^-} u(r \cdot e^{i\theta_0}) = U(\theta_0).$$

(Read in Ahlfors. Schwarz Theorem) <u>Ch4</u>. last section.

and $u(z)$ is a harmonic function.

- <u>Mean value property</u>: (easy consequence of CIF).
- (hol'c function:) say f is hol'c function in Ω.

$z_0 \in \Omega, \qquad \overline{D_r(z_0)} \subset \Omega.$



then: $\quad f(z) = \sum_{n=0}^{\infty} a_n \cdot (z - z_0)^n.$

$$a_0 = f(z_0), \quad a_1 = f'(z_0), \quad a_2 = f^{(2)}(z_0)/2! \;\cdots$$

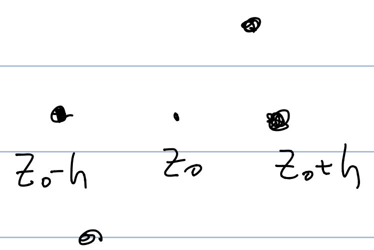
$$a_n := \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(z_0 + r \cdot e^{i\theta}) \cdot \frac{e^{-in\theta}}{r^n} \cdot d\theta$$

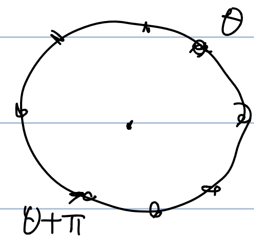
in particular

$$a_0 = f(z_0) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

(Also true for harmonic functions.)

$$a_1 = f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0-h)}{2h}$$





$$\frac{1}{2\pi} \int f(z_0 + r \cdot e^{i\theta}) \cdot \frac{e^{-i\theta}}{r}\, d\theta$$

Homework #7

(1) if $f: D \to D \quad f(0) = 0.$

$= \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim \frac{f(z)}{z}$

consider $\frac{f(z)}{z}$, i.e. $\quad F(z) = \begin{cases} f'(0) & z = 0 \\ \frac{f(z)}{z} & z \neq 0 \end{cases}$

[illegible]