- Midterm : Nov 10th in class. same format as last time.
- Today : 1. HW #5.

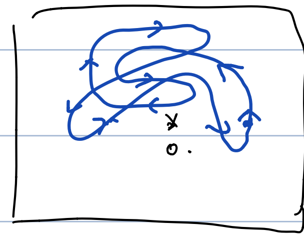
  2. Harmonic Function.

  3. Review. Ch 3.

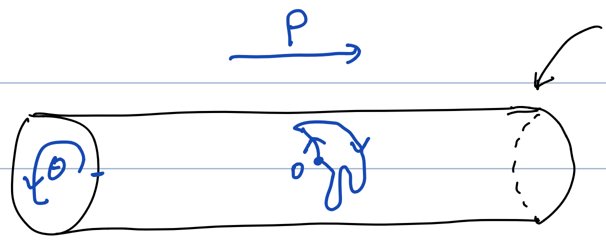
$\gamma(0)=\gamma(1)=1$. $\quad \mathbb{C}^* = \mathbb{C}\setminus\{0\}$.

1. let $\gamma : [0,1] \to \mathbb{C}^*$, and $\int_\gamma \frac{1}{z}\,dz = 0$.

namely $n(\gamma, 0) = 0$



Claim : γ can be homotoped to a constant curve.



$\mathbb{C}^* \simeq \mathbb{R}\times S^1$

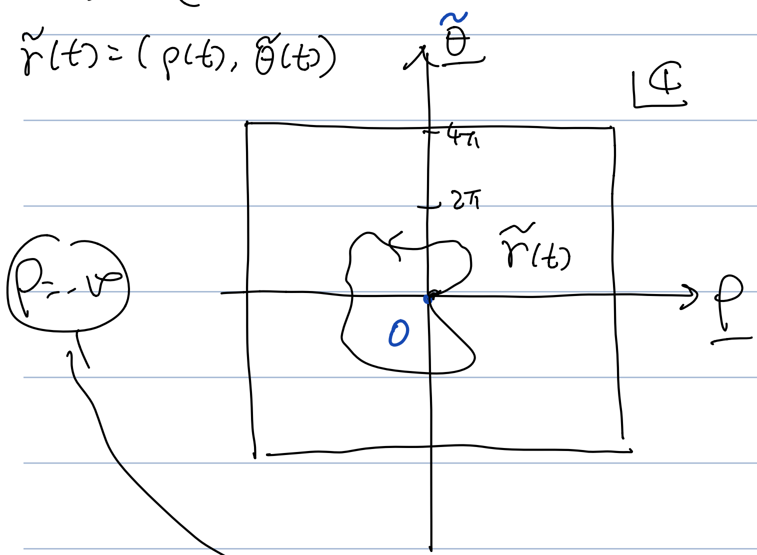
$z = e^{\rho + i\theta}$

$\rho \in \mathbb{R}, \quad \theta \in S^1$ $\quad [0, 2\pi]$ $\quad 0\sim 2\pi$.

the winding number around 0 = the winding number around the cylinder.

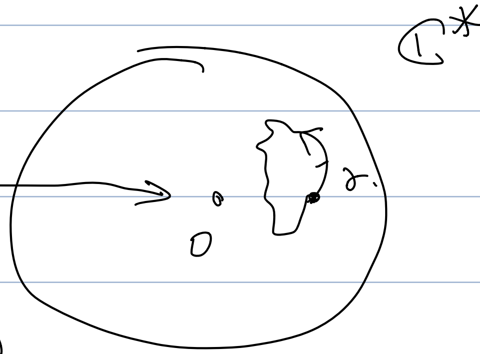
$\gamma(t) = e^{\rho(t) + i\theta(t)}$.

$\tilde{\gamma}(t) = (\rho(t), \tilde{\theta}(t))$



in this $(\rho, \tilde{\theta})$ plane, one can use linear interpolation from $\tilde{\gamma}$ to a constant curve.

exp.



$\gamma(t) = e^{\rho(t) + i\theta(t)}$

$n(\gamma, 0) = 0$ guarantees that $\theta(0) = \theta(1)$. #.

2: Harmonic Function on $\mathbb{D}$.

recall:

- Def: $u: \mathbb{D} \to \mathbb{R}$. $u(x,y)$.
- $\partial_x^2 u + \partial_y^2 u = 0$.

Ex: $u = ax + by$

$u = a(x^2-y^2) + bxy$.

$u = a \cdot (x^3 - 3xy^2) + b(y^3 - 3x^2y)$.

......

- In general, if $f: \mathbb{D} \to \mathbb{C}$ hol'c then $\underline{u = \mathrm{Re}(f)}$ is harmonic.

- conversely, if u is harmonic, then $\exists\ f: \mathbb{D} \to \mathbb{C}$, $\underline{\text{hol'c}}$ s.t. $\mathrm{Re}(f) = u$.

  $f = u + iv$, v is determined upto a const.

- Suppose u can be extended continuously to the boundary circle, then u can be recovered from its boundary value.

- Dirchlet boundary condition for Laplace equation:

$$\begin{cases} \Delta u = 0 & \text{on } \mathbb{D} \\ u|_{\partial \mathbb{D}} = u_0 & \text{on } \partial \mathbb{D} \end{cases} \Rightarrow \exists! \text{ sol'n}.$$

u_0 given continuous function on the boundary. (regularity condition can be relaxed)

- Explicit formula to go from boundary value of $\mathcal{U}$ to u: Poisson <u>kernel</u>:

$$P(z, \theta)$$

output index → z; θ ← input index

Analogy for matrix. M_{ij}, $n \times n$.

$$(v_j) \mapsto \sum_{j=1}^{n} M_{ij} v_j = (w_i)$$

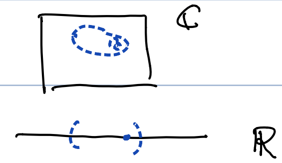
value in the interior. ↘ boundary value.

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P(z,\theta) \cdot \mathcal{U}(\theta) \cdot d\theta$$

$$P(z,\theta) = \mathrm{Re}\left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right) \quad \text{(upto sign).}$$

(Schwarz thm. <u>Ahlfors</u>. Ch4. section 6)

- For hol'c f, we have open mapping thm: hol'c $f: \Omega \to \mathbb{C}$ $\Rightarrow$ $\forall\, U \subset \Omega$ open. $f(U)$ is open.
- $u = \mathrm{Re}(f)$: $\mathbb{D} \xrightarrow[\text{open map}]{f} \mathbb{C} \xrightarrow[\text{open map.}]{\mathrm{Re}} \mathbb{R}$



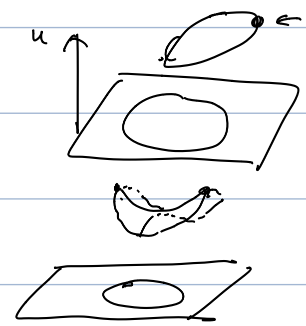
then if $\mathcal{U} \subset \mathbb{D}$ is open. $f(\mathcal{U})$ is an open set, then $\mathrm{Re}(f(\mathcal{U}))$ is open.

<u>Thus</u>, harmonic functions are open. maps: $\mathbb{C} \to \mathbb{R}$.

- <u>Maximum principle</u>: u has no max achieved in ~~the~~ $\mathbb{D}$.

<u>Ex</u>: . u is linear, $u = x$.

. u is quadratic, $u = x^2 - y^2$



<u>Cor</u> (of max principle), if $u: \mathbb{D} \to \mathbb{R}$. harmonic, u is continuous on $\overline{\mathbb{D}}$, then. u achieve the $\sup_{z \in \mathbb{D}} u(z)$, on the boundary. $\overline{\mathbb{D}} \setminus \mathbb{D}$. <u>In general</u>, one can replace $\mathbb{D}$ by any bounded (~~simply connected~~) region. (see Stein for the corollary

of holomorphic function).

- Relation between <u>Powers Series</u> and <u>Fourier series</u>:

<u>first</u> do it for holomorphic function.,

- holomorphic function $f: \overline{\mathbb{D}} \to \mathbb{C}$ (i.e. f is hol'c in some nbhd of $\overline{\mathbb{D}}$)

- $f(z) = \sum_{n=0}^{\infty} a_n \cdot z^n$, $|z| \leq 1$.

in particular. if $|z| = 1$, $z = e^{i\theta}$.

$$f(e^{i\theta}) = \sum_{n=0}^{\infty} a_n \cdot e^{in\theta}.$$

This is the Fourier expansion of $f: \underset{\partial\mathbb{D}}{\overset{S^1}{\shortparallel}} \to \mathbb{C}$

This can be used to go from boundary value to the interior val.

<u>e.g</u>: if we know

$$f(e^{i\theta}) = 3 \cdot e^{i\theta} + 5 \cdot e^{i2\theta}$$

then we know

$$f(r \cdot e^{i\theta}) = 3 \cdot r \cdot e^{i\theta} + 5 \cdot r^2 \cdot e^{i2\theta}$$

i.e. $f(z) = \underline{3z + 5z^2}$.

<u>Key</u>: $\overset{\underline{\forall n \geq 0}}{e^{in\theta}}$ on the $\partial\mathbb{D}$ can be extended to $\underline{z^n}$ in $\mathbb{D}$,

"positive frequency Fourier mode"

- <u>Now</u>, say $u = \underset{f(z) + \overline{f(z)}}{\overset{}{=}}\ 2\,\mathrm{Re}(f)$.

how to recover u from its boundary value $u(\theta)$ to interior. using Fourier modes of $u(\theta)$.

$$\mathcal{U}(\theta) = \sum_{n\in\mathbb{Z}} c_n \cdot e^{in\theta} \qquad c_n \in \mathbb{C}.$$

$\because$ $\mathcal{U}(\theta)$ is real valued

$$\therefore \quad \mathcal{U}(\theta) = \overline{\mathcal{U}(\theta)}$$

$$\sum_n c_n \cdot e^{in\theta} = \overline{\sum_n c_n \cdot e^{in\theta}} = \sum_{n\in\mathbb{Z}} c_n^* \cdot e^{-in\theta}.$$

$$= \sum_{n\in\mathbb{Z}} c_{-n}^* \cdot e^{in\theta}$$

$$\Rightarrow \quad \underline{c_n = c_{-n}^*}$$

(e.g. $\sin(\theta) = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \left(\dfrac{-1}{2i}\right)\cdot e^{-i\theta} + 0 + \left(\dfrac{1}{2i}\right)\cdot e^{i\theta}$

complex conjugate of each other.

So, we can define

$$f(z) = \tfrac{1}{2} c_0 + \sum_{n\geq 1} c_n \cdot z^n.$$

then

$$\overline{f(z)} = \tfrac{1}{2} c_0 + \sum_{n\geq 1} c_n^* \cdot \overline{z}^n.$$

$$f(z) + \overline{f(z)}\Big|_{z=e^{i\theta}} = c_0 + \sum_{n\geq 1} c_n \cdot e^{in\theta} + \sum_{n\geq 1} \underbrace{c_n^*}_{= c_{-n}} \cdot e^{-in\theta}$$

$$= \sum_{n\in\mathbb{Z}} c_n \cdot e^{in\theta} = \mathcal{U}(\theta).$$

we can define

$$u(z) = f(z) + \overline{f(z)} = 2\cdot \mathrm{Re}(f(z)). \qquad \#$$

$\equiv$

<u>Review of Ch3:</u>

- Consequence of Cauchy integral formula.
  - Residue thm:
    - definition of residue at a point z_0 is

the coefficient of $\frac{1}{z-z_0}$ in the Laurent expansion at z_0,

(say : f has a pole at z_0 of order n, then we can write

$$f(z) = \frac{h(z)}{(z-z_0)^n} \qquad h(z) \text{ is hol'c at } z_0.$$

$$= \frac{\sum_{i=0}^{\infty} a_i \cdot (z-z_0)^i}{(z-z_0)^n} = \frac{a_0}{(z-z_0)^n} + \frac{a_1}{(z-z_0)^{n-1}} + \cdots$$

$$+ \cdots + \frac{a_{n-1}}{z-z_0} + \cdots$$

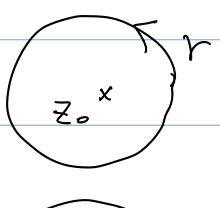
plugin the Taylor expansion of h, then cancel out common factors)

- If γ is a simple closed curve, f: is meromorphic with finitely many poles inside γ, then

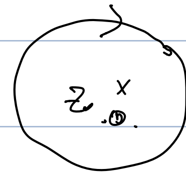
$$\int_\gamma f(z)\, dz = 2\pi i \left(\sum_{\substack{p:\text{ residue} \\ \text{of } f \text{ insider}}} \mathrm{Res}_p\, f(z) \right).$$

in Ahlfors.

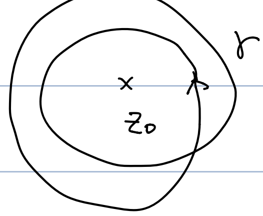
- Winding number + argument principle.



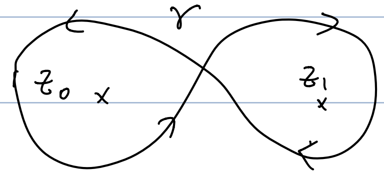
$n(\gamma, z_0) = 1$



$n(\gamma, z_0) = -1$



$n(\gamma, z_0) = 2.$



$n(\gamma, z_0) = 1$

$n(\gamma, z_1) = -1.$

Def:

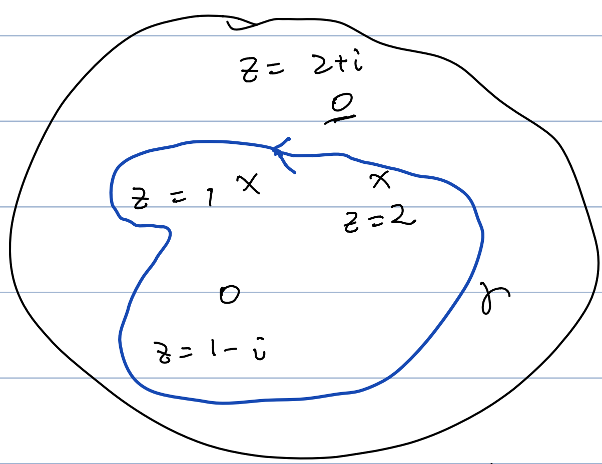
$$n(\gamma, z_0) = \frac{1}{2\pi i}\int \frac{1}{z - z_0}\, dz$$

· Say $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ rational function,

γ : is a closed curve in $\mathbb{C}$ that doesn't contain any zero or pole of f on γ.

then
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = (\#\text{ zero inside } \gamma) - (\#\text{ poles inside } \gamma).$$

Ex:



o : zero

x : pole.

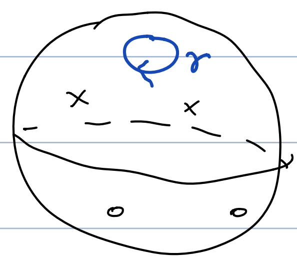
$$f(z) = \frac{(z-(2+i))(z-(1-i))}{(z-1)(z-2)}$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = 1 - 2 = -1.$$

$$\frac{f'(z)}{f(z)} = \frac{(-1)}{\underset{\sim}{z-1}} + \frac{(-1)}{\underline{z-2}} + \frac{(+1)}{\underline{z-(1-i)}} + \underbrace{\cdots}_{\text{hol'c inside } \gamma.}$$

$f(z) = z$, $\gamma =$ ⟲ unit circle.

- In particular, if f is a rational function, then the (total # of zero) = (total # of poles).



- <u>Rouché thm</u> : $(\#\text{ zero of } f \text{ inside } \gamma) = (\#\text{ zero of } \underline{f+g} \text{ inside } \gamma)$

  <u>if</u> $|g| < |f|$ on γ.

  i.e. if we "turn on" the perturbation g,
  try $f_\varepsilon(z) = f(z) + \varepsilon \cdot g(z)$, $\varepsilon \in [0,1]$,
  then the <u># of zero of f_ε inside γ</u> remain const w.r.t. ε.

- <u>Open</u> mapping & maximum modulus principle.

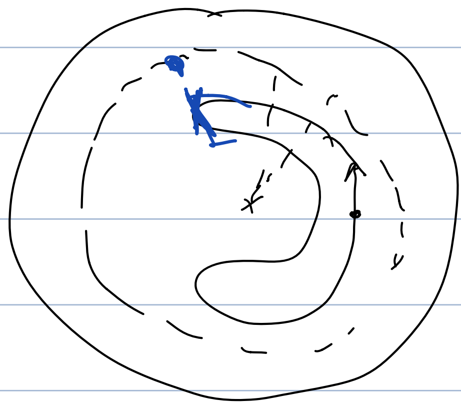
- Review Suggest :
  - practical computation, try Schaum's outline

- conceptual reinforcement, try Ahlfors. Ch 4.

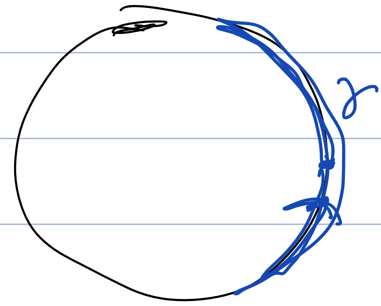
- Alex's sol'n :

"project"

push the curve to unit circle



$\longrightarrow$



$$\gamma(t) = e^{i\theta(t)} \qquad t \in [0,1],$$

now $\theta_s(t) = (1-s)\,\theta(t).$ $\quad s \in [0,1]$

so $\theta_0(t) = \theta(t)$

$\theta_1(t) = 0.$