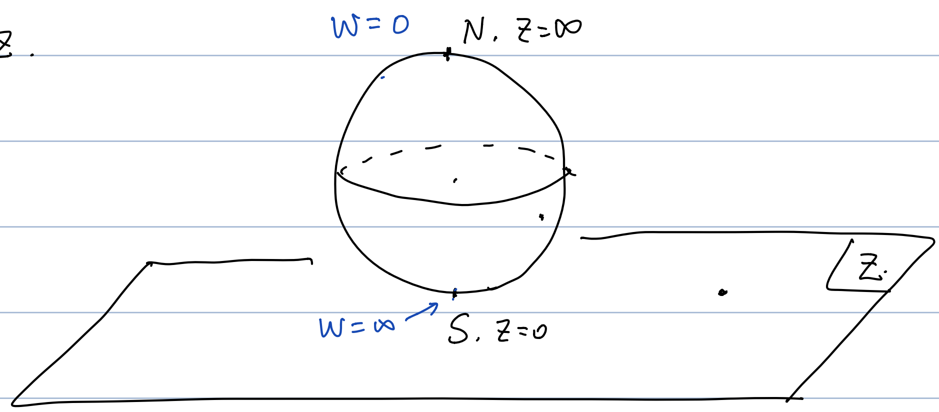
Today: Stein Ch 3. Section 3 & 4.

- meromorphic function on $\hat{\mathbb{C}}$ is rational.
- argument principle. "$\arg(z)$"
- Rouché theorem.
- $\ast$ open mapping. maximum modulus thm..

- A meromorphic function f on an open set Ω, is a function that has a sequence of poles in Ω, (possibly infinite), and these poles ~~does~~ not have an accumulation point.

- $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

  use a new complex coordinate near $z = \infty$, by letting $w = 1/z$.



- f is a meromophic function on $\hat{\mathbb{C}}$ means.
  - $f(z)$ restricted on $\mathbb{C} \subset \hat{\mathbb{C}}$ is a meromophic function.
  - $\tilde{f}(w)$ restricted on. $\mathbb{C}_w \subset \hat{\mathbb{C}}$, is a meromophic function

    " $f(z) = f(1/w)$. " $\hat{\mathbb{C}} \setminus \{w = \infty\}$.

Thm: if f is a meromorphic function on $\hat{\mathbb{C}}$,

then f is a rational function. i.e.

$f(z)$ on $\mathbb{C}$ can be written as $\frac{P(z)}{Q(z)}$ for

P, Q polynomials

Before we prove thm, some examples.

	zeros	poles
$f(z) = \frac{1}{z}$ $\tilde{f}(w) = w$	$\{z = \infty\}$ $\{w = 0\}$	$\{z = 0\}$ $\{w = \infty\}$
$f(z) = z(z+1)$ $\tilde{f}(w) = \frac{1}{w}\left(\frac{1}{w}+1\right) = \frac{1+w}{w^2}$	$z = 0, -1.$	a pole of order 2 at $w=0$ (or. $z=\infty$).
$f(z) = \frac{z+1}{z+2}$	$z = -1,$	$z = -2$

no pole or zero at $z=\infty$

$\because \lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{1+\frac{1}{z}}{1+\frac{2}{z}} = 1.$

$\tilde{f}(w) = \frac{1+w}{1+2w}$

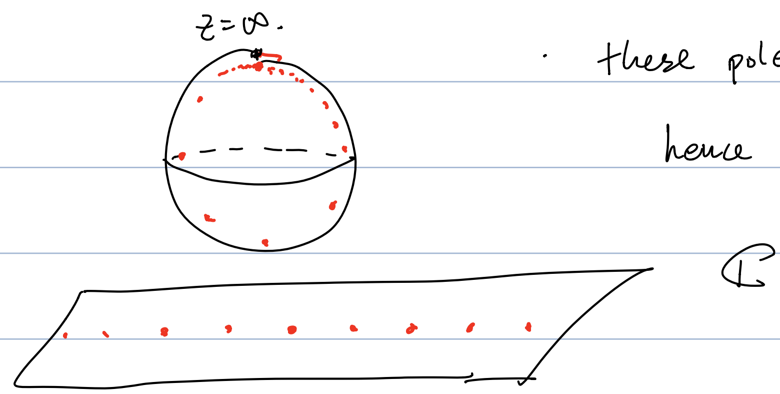
$\tilde{f}(0) = 1.$

non-example for meromorphic function over $\mathbb{C}$ but ~~over~~ not over $\hat{\mathbb{C}}$

$f(z) = \frac{1}{\sin(z)}$

- poles at $z = n\cdot\pi$.
- these poles do not accumulate over $\mathbb{C}$.
- these poles converge to ∞ in $\hat{\mathbb{C}}$.

hence $f(z)$ is not meromorphic in $\hat{\mathbb{C}}$.



$\tilde{f}(w) = \frac{1}{\sin\left(\frac{1}{w}\right)}$

$w=0$ is an essential singularity.

Pf. of thm: <u>step 1</u>: f has finitely many poles in $\hat{\mathbb{C}}$.

<u>step 2</u>: say $z_1, \cdots, z_n$ are the poles,

we define $f_j(z)$ to the "singular part" of f

at z_j, then we consider

$$R(z) = f(z) - \sum_{j=1}^{n} f_j(z)$$

then conclude $R(z)$ is a constants.

(1) Recall, a set is sequentially compact if any infinite sequence of points. in this set has accumulation point.

Since $\hat{\mathbb{C}} \simeq S^2$ is compact, we cannot have ∞ many poles.

(2) Let's consider first. the possible poles at $z=\infty$. (or $w=0$).

$$\tilde{f}(w) = f\left(\frac{1}{w}\right) = \underbrace{\frac{b_n}{w^n} + \frac{b_{n-1}}{w^{n-1}} + \cdots + \frac{b_1}{w^1}}_{\tilde{f}^{sing}_{w=0}(w)} + \tilde{R}_0(w)$$

$\tilde{f}(w) - \tilde{f}^{sing}_{w=0}(w)$ is now a regular function. near $w=0$.
holomorphic = regular

$\tilde{f}^{sing}_{w=0}$ in z coordinate is

$$\cdot\ f^{sing}_{z=\infty}(z) = \tilde{f}^{sing}_{w=0}\left(\frac{1}{z}\right) = \frac{b_n}{(1/z)^n} + \cdots + \frac{b_1}{(1/z)}$$

$$= b_n \cdot z^n + b_{n-1} \cdot z^{n-1} + \cdots + b_1 \cdot z$$

Now for other poles for $z \in \mathbb{C}$.

expand near $z = z_k$

$$f(z) = \underbrace{\frac{(\cdots)}{(z-z_k)^{n_k}} + \frac{(\cdots)}{(z-z_k)^{n_k-1}} + \cdots + \frac{(\cdots)}{(z-z_k)^1}}_{f_k(z) = f^{sing}_{z=z_k}(z)} + \tilde{R}_k.$$

Consider $R(z) = f(z) - f^{sing}_{\infty}(z) - f^{sing}_{z_1}(z) - \cdots - f^{sing}_{z_k}(z)$.

(Claim:) $R(z)$ near $z=\infty$.

$R(z)$ is bounded near each of the poles, hence by Liouville thm, $R(z) = C$, constant.

Then. $f(z) = \left(b_n z^n + \cdots + b_1 z^1\right) + \frac{(\cdots)}{(z-z_1)^{n_1}} + \cdots + \frac{(\cdots)}{(z-z_1)^1}$

$$+ f^{sing}_{z_2}(z) + \cdots + f^{sing}_{z_k}(z) + C$$

is a finite sum of rational function, hence by clearing denominator, we can write.

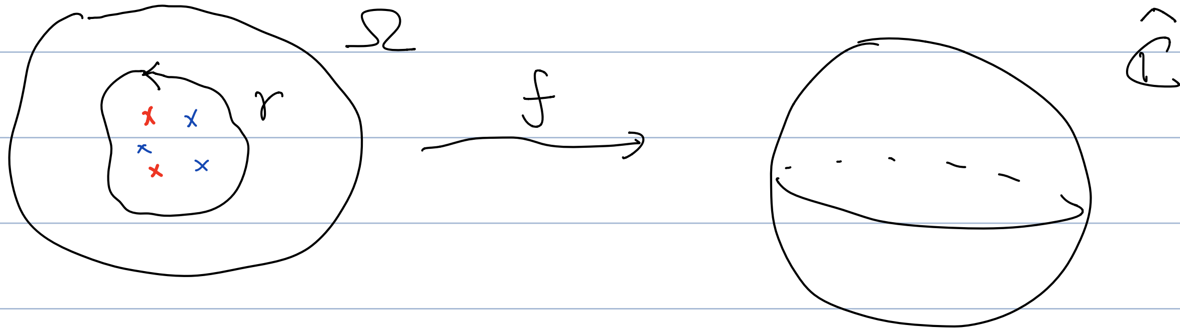
$$f(z) = \frac{P(z)}{(z-z_1)^{n_1}(z-z_2)^{n_2}\cdots(z-z_k)^{n_k}} \qquad \#.$$

Next, we move to section (4).

Argument Principle:

× : pole

× : zero.



consider $f : \Omega \to \hat{\mathbb{C}}$ a meromophic function.

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz.$$

for simplicity, consider γ to be a simply closed curve.
(e.g. $\Omega = D_{1+\varepsilon}(0)$, $\gamma = C$ unit circle).

- $\underbrace{\frac{f'(z)}{f(z)}\, dz}_{\text{we didn't define yet}} = \frac{d\, f(z)}{f(z)} = $ " $d\,(\log f(z))$ ".

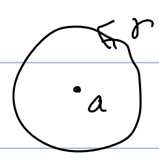
$\log f(z) = \log |f(z)| + i \cdot \arg \cdot \underline{f(z)}$

but <u>argument</u> of a complex number is ill-defined.,
its value is ambiguous upto $2\pi \cdot n$.

However, $d\, \underline{\log f(z)}$ is well-defined.

<u>Ex</u>: • if $f(z) = z - a$, $f'(z) = 1$., we have.

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 2\pi i \cdot 1.$$



- if $f(z) = \underline{5}(z-a)$, $f'(z) = 5$.

$$\int_\gamma \frac{5}{5(z-a)} \cdot dz = \int_\gamma \frac{1}{\underline{z-a}}\, dz = 2\pi i.$$

Claim : if $f(z)$ has an order n zero at z_0,

then near z_0,

$$\frac{f'(z)}{f(z)} = \frac{n}{z-z_0} + \text{regular part.}$$

(ex, $f(z) = z^n$, $\frac{f'(z)}{f(z)} = \frac{n \cdot z^{n-1}}{z^n} = \frac{n}{z}$)

Pf : in general,

$$f(z) = (z-z_0)^n \cdot g(z).$$

$\uparrow$ non-vanishing near z_0.

$$f'(z) = \left((z-z_0)^n\right)' \cdot g(z) + (z-z_0)^n \cdot g(z)'$$

$$= n \cdot (z-z_0)^{n-1} \cdot g(z) + (z-z_0)^n \cdot g(z)'$$

then $$\frac{f'(z)}{f(z)} = \frac{n \cdot (z-z_0)^{n-1} \cdot g(z) + (z-z_0)^n \cdot g'(z)}{(z-z)^n \cdot g(z)}$$

$$= \frac{n}{z-z_0} + \frac{g'(z)}{g(z)} \qquad \#,$$

$\uparrow$ holomorphic near z_0.

Claim #2 : if f has a pole of order n at z_0,

then

$$\frac{f'(z)}{f(z)} = \frac{-n}{z-z_0} + (\text{regular.})$$

e.g. $f(z) = \frac{1}{z^2}$, $f'(z) = \frac{-2}{z^3}$, $\frac{f'(z)}{f(z)} = \frac{-2}{z}$. .... !

Thm 4.1 . Suppose $f : \Omega \to \hat{\mathbb{C}}$ is meromophic,

suppose

$\gamma \subset \Omega$ is a simple closed curve, interior of γ

is also in Ω. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = (\text{number of zero inside } \gamma) - (\text{number of poles inside } \gamma).$$

where the number takes into account multiplicity.

Pf: $\underbrace{\frac{f'(z)}{f(z)}}$ subtract the $\underline{\text{singular part of } f'/f}$

at zero and poles of $f(z)$, will be a regular function inside γ, hence the integral will be zero.

Hence,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \underbrace{\frac{a_1}{z-\alpha_1}+\cdots+\frac{a_n}{z-\alpha_n}}_{\text{contribution of zero of } f(z)} + \frac{-b_1}{z-\beta_1}+\cdots+\frac{-b_m}{z-\beta_m}\,dz.$$

$f(z)$ has zeros $\alpha_1, \cdots, \alpha_n$ inside γ

with order $a_1, \cdots, a_n$.

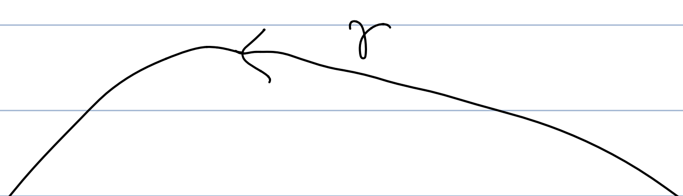
has poles $\beta_1, \cdots, \beta_m$ inside γ.

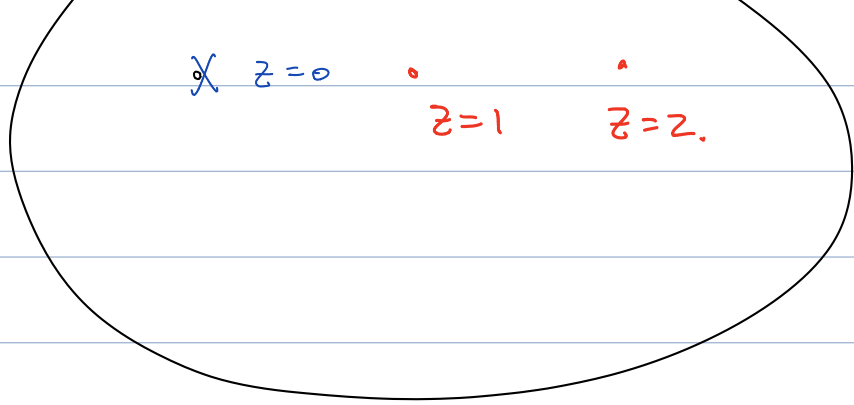
with order $b_1, \cdots, b_m$.

$$= a_1 + \cdots + a_n - b_1 - b_2 - \cdots - b_m$$

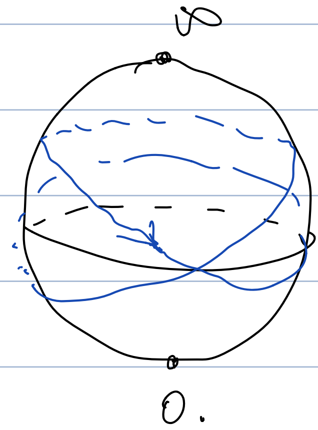
#.

Ex: $f(z) = \dfrac{z}{(z-1)(z-2)}$





$\searrow f$



as z traverse γ.

$f(z)$ tranverse a curve $f(\gamma)$. in $\hat{\mathbb{C}}$,

which wraps around $\underline{0}$ once.

and wraps around $\underline{\infty}$ twice.

Let $w = f(z)$.

$$\frac{1}{2\pi i}\int_{f(\gamma)} \frac{dw}{w} = \frac{1}{2\pi i}\int_{f(\gamma)} d\,\underline{\log w}$$

= counts the winding number around 0