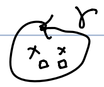
Recall : argument principle = " winding number of the image curve $f(\gamma)$ around 0 "

(1)

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\, dz = \#\text{ of zero of } f - \#\text{ of pole of } f.$$

↑ simple closed curve



$\times$: pole

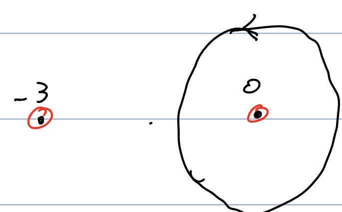
$\square$: zero.

(2) Rouché theorem : comparison of the number of zeros of 2 " nearby " holomorphic functions inside γ.

Consider $f(z) = z^2 + 3z$, inside the unit circle C

How many zeros are there of f inside C ?

$f(z) = z(z+3)$, has roots at 0 and -3.



Consider modification of $f(z)$. $\quad /\!/ \; (z+\frac{3}{2})^2 + t - (\frac{3}{2})^2$

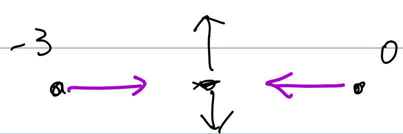
$t \in \mathbb{C}$

Ex : $f_t(z) = z^2 + 3z + t$. $\quad |t|$ is small.

now the roots of $f_t(z)$ depends on t.

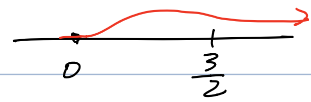
as t increases from 0 to $\underset{\sim}{(\frac{3}{2})^2}$, then past, the roots

moves as

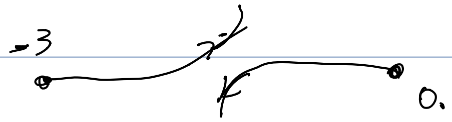


$$z_{\pm}(t) = \pm\sqrt{\left(\tfrac{3}{2}\right)^2 - t} - \tfrac{3}{2}.$$

If t moves like



then the 2 roots of $f_t(z)$ moves like



<u>Take away</u>: • if $|t|$ is small, then the roots doesn't move much, and the number of zeros inside C remains unchanged.

<u>Ex2</u>: same f and C as above.

$$f_t(z) = \underbrace{t \cdot z^{10}} + \underbrace{z^2 + 3z}_{f(z).}$$

$|t|$ small.

is it still true that for small $|t|$, there is only one root inside C?

• as soon as $\underline{t \neq 0}$, the total number of roots of $f_t(z)$ is $10 = \deg$ of $f_t(z)$.

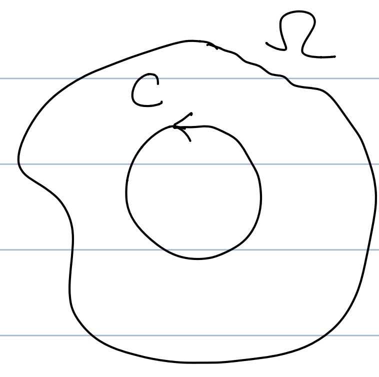
Heuristic $|z^{10}|$ is small when $|z| < 1$.

so, the new roots will occur far away.

· $f_t(z)$ is a function close to $f(z)$, not everywhere but at least on any compact subset K, when t is small.

$$\| f_t - f \|_K := \sup_{z \in K} | f_t(z) - f(z) | \longrightarrow 0$$
as $t \to 0$.

<u>Rouché theorem</u>: Let f and g be hol'c functions in Ω on Ω, and let C be a circle (or any simple closed curve) with its interior contained in Ω, $f(z) \neq 0$ for $z \in C$.



If $|f(z)| > |g(z)| \quad \forall z \in C$

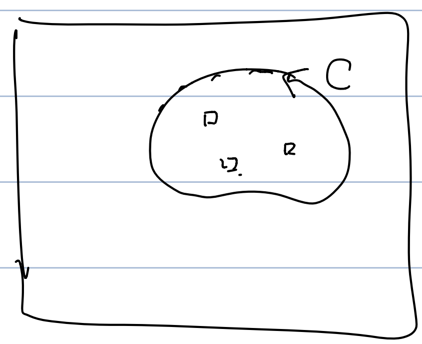
then.

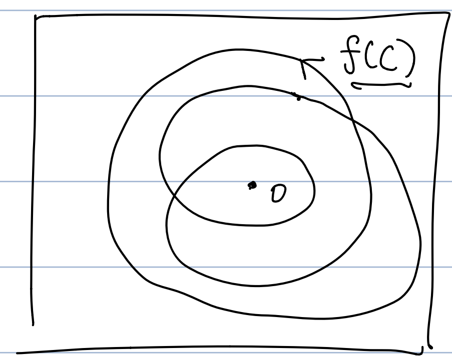
\# zero of f inside C

$=$ \# zero of $f+g$ inside C.

<u>Intuition</u>: (1) · \# zero of f inside C

$=$ the winding number of $f(C)$ around 0.





$\because$ there is no zero of f <u>on</u> C

$\therefore$ $f(C)$ does not pass through zero.

$$\frac{1}{2\pi i}\oint_{z\in C}\frac{f'(z)\,dz}{f(z)} = \frac{1}{2\pi i}\int_{w\in f(C)}\frac{dw}{w}.$$

(2) f can be deformed to $f+g$ by

$$f_t(z) = f(z) + t\cdot g(z). \qquad t\in[0,1]$$

consider how the image curve $f_t(C)$ deforms.

key point: $f_t(C)$ never pass through zero. as t varies. Hence the winding number doesn't change.

$f_t(C)$ never pass through zero

$\iff \forall t\in[0,1],\ \forall z\in C,\quad f_t(z)\neq 0.$

$\iff \forall t,\ \forall z\in C \quad f(z)+t\cdot g(z)\neq 0.$

$\Leftarrow \forall t,\ \forall z\in C \quad |f(z)| > t\cdot|g(z)|$

$\Leftarrow \forall z\in C \quad |f(z)| > |g(z)|$, which is given.

#.

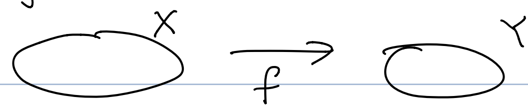
Open Mapping Theorem:

- Recall if $f: X\to Y$ is a continuous map between two topological space, f is open if and only if for all $U\subset X$ open set, $f(U)$ is open.

"f is an open map if f sends open set to open set".

distinction between f is continuous and f is open

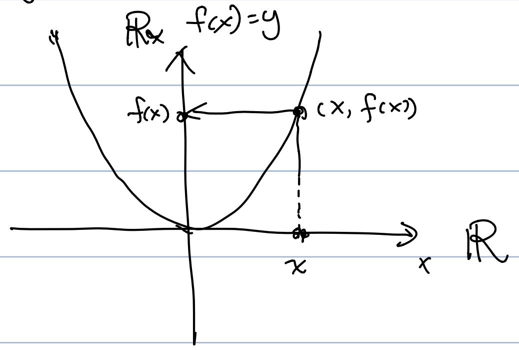
- f is continuous :

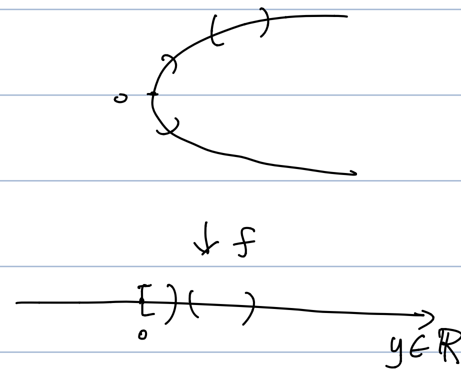


$\forall\, V \subset Y$ open, $f^{-1}(V)$ needs to be open.

Thm : if $f : \Omega \to \mathbb{C}$ is a non-constant hol'c function, then f is an open map.

Ex : (1) $f : \mathbb{R} \to \mathbb{R}$. $f(x) = x^2$.





① identify the domain of f with the graph of f

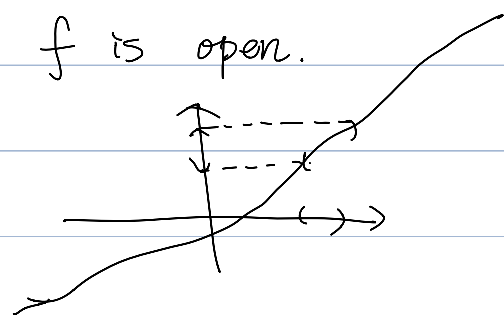
② then the map f is just the projection of the graph of f to the y- axis.

this is not an open map, because the open interval
$f : (-r, r) \longmapsto [0, r^2)$

$\uparrow$ not an open interval

(2). Consider $f : \mathbb{R} \to \mathbb{R}$. if $f'(x)$ is never zero, (hence has constant sign), then f is open.

(using implicit function theorem.)

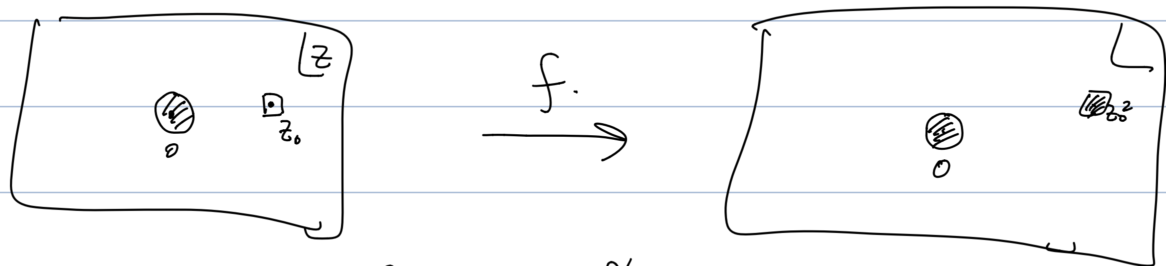


$f: \mathbb{R} \to \mathbb{R}$,

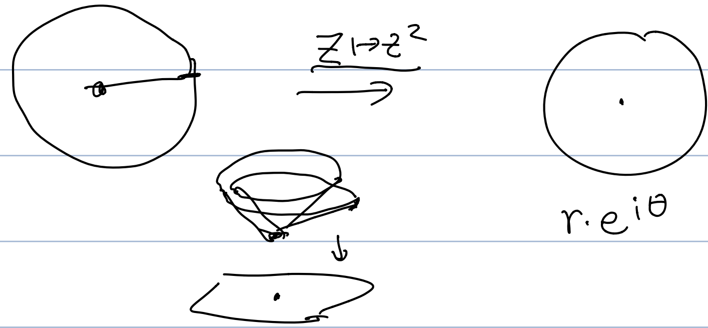
(3) $f(x) = \text{const}$ is not open, the image is a closed pt in $\mathbb{R}$.

Why complex number saves the day?

Ex: $f(z) = z^2$.



- if $z_0 \neq 0$, $f'(z_0) = 2z_0 \neq 0$.
  hence a small disc around z_0 maps to a small open around $f(z_0)$.
- if $z_0 = 0$, $f'(z_0) = 0$, then the disk $D_\varepsilon(0) \longrightarrow f(D_\varepsilon(0))$ as a 2-to-1 cover. (except at 0).



$r \cdot e^{i\theta} \mapsto r^2 \cdot e^{i \cdot 2\theta}$.

Pf: We need to show that, for any $U \subset \Omega$ open, $f(U)$ is open.

$\iff$ $\forall\, w_0 \in f(U)$, there exists an open nbhd of w_0 inside $f(U)$

$\iff$ $\forall\, w_0 \in f(U)$, there exists a $\delta > 0$. small enough, s.t. $D_\delta(w_0) \subset f(U)$.

Pick a point $z_0 \in U$, s.t. $f(z_0) = w_0$.

Consider Taylor expansion around z_0.

i.e. $\exists\ \varepsilon > 0$, small enough s.t. $\forall\ |z - z_0| < \varepsilon$,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + f''(z_0) \cdot \frac{(z-z_0)^2}{2!} + \cdots$$

$$= w_0 + \underbrace{f^{(n)}(z_0)}_{f^{(n)}(z_0) \neq 0} \frac{(z-z_0)^n}{n!} + f^{(n+1)}(z_0) \frac{(z-z_0)^{n+1}}{(n+1)!} + \cdots$$

$$= w_0 + (z - z_0)^n \cdot h(z)$$

$h(z)$ is hol'c. for $|z - z_0| < \varepsilon$.

$h(z_0) \neq 0$.

(compare with §3.1).

By shrinking ε, we may assume $h(z) \neq 0$ for $|z - z_0| \leq \varepsilon$.

Claim: if $\underset{\text{distance between } w_0 \text{ and the image curve.}}{\delta} := \inf\limits_{z:\ |z-z_0| = \varepsilon} \left(|f(z) - w_0| \right) > 0$

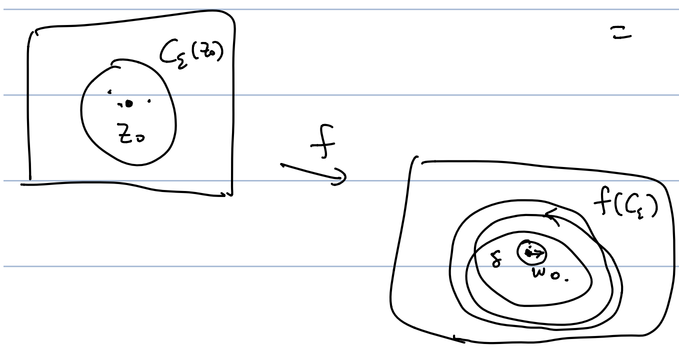
indeed.

$$= \inf_{|z - z_0| = \varepsilon} |z - z_0|^n \cdot |h(z)|$$

$$= \varepsilon^n \cdot \underbrace{\inf_{|z-z_0| = \varepsilon} |h(z)|}_{> 0} > 0.$$

[$\because$ we are taking inf of a continuous function over a compact set. $C_\varepsilon(z_0)$.

$\therefore$ the inf is achieved at some point $C_\varepsilon(z_0)$]



claim:

Finally, for any w, s.t. $|w - w_0| < \delta$, there exist some z, s.t. $|z - z_0| < \varepsilon$, with $f(z) = w$.

consider $f(z) - w$. for $z \in D_\varepsilon(z_0)$.

$$f(z)-w = (f(z)-w_0) - (w-w_0).$$

$\because$ $|f(z)-w_0| \geq \delta > |w-w_0|$. $\forall z \in C_\varepsilon(z_0)$.

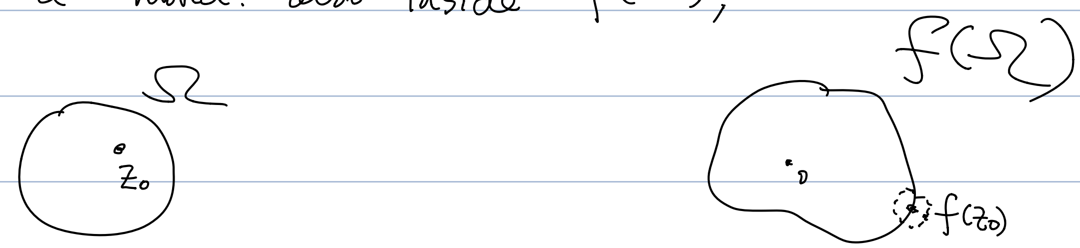
$\therefore$ $\underline{f(z)-w_0}$ and $\underline{f(z)-w}$ have the same number of zeros inside $C_\varepsilon(z)$, by Rouché thm.

$\therefore$ $f(z)-w$ has a zero inside $C_\varepsilon(z_0)$. finishing the claim. #.

Thm: (Maximum modulus Principle) $\Omega \subset \mathbb{C}$ open.
If $f: \Omega \to \mathbb{C}$ is a non-constant hol'c function, then there is $\underline{\underline{\text{no}}}$ $z_0 \in \Omega$, s.t.

$$|f(z_0)| = \sup_{z\in\Omega} |f(z)|.$$

Pf: Assume there is such a z_0, then w_0 $f(z_0)$ has a nbhd $D_\varepsilon(w_0)$ also inside $f(\Omega)$,

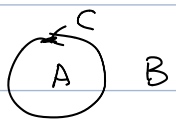


but there are points $w \in D_\varepsilon(w_0)$, s.t. $|w| > |w_0|$. #.

HW 6 #5 : $C = \{|z| = 1\}$ unit circle

- $g : C \to \mathbb{C}$ ~~contin~~ smooth, function on C.

- $$f(z) := \frac{1}{2\pi i} \int_C \frac{g(w)}{w - z}\, dw.$$ for $z \in \{|z|<1\} = A$ and $z \in \{|z|>1\} = B$



Q1: (1) this integral make sense for region A, B.

(2). For any $z_0 \in C$, let $z = r \cdot z_0$ for $r \in (0,1)$.

$$\lim_{r \to 1} f(r \cdot z_0) = g(z_0).$$ is it true ?

Hint : consider the example: $g(e^{it}) = e^{int}$ for $n \in \mathbb{Z}$. , and see what output f you get.

HW6

- e^z , as $|z| \to \infty$ along a ray $z = re^{i\theta}$, for fixed θ, for $r \to \infty$.

$$e^{r \cdot e^{i\theta}} = e^{r\cos\theta + i r \cdot \sin\theta} = \underbrace{e^{r\cos\theta}}_{\text{modulus}} \underbrace{e^{ir\sin\theta}}_{\text{phase}}.$$

e.g $\theta = \frac{\pi}{2}$, $\sin\theta = 1$. phase $= e^{ir}$

- if $\cos\theta = 0$, then modulus remains $=1$, phase is rotating, as $r \to \infty$

- if $\sin\theta = 0$, then phase is fixed. and positive,
  $e^{r\cos\theta} \to 0$ or ∞ depending on sign of $\underline{\cos\theta}$
  (as $r \to \infty$)

- Laurent expansion of a meromorphic function near a pole :

(try them in §3.1). if z_0 is an order n pole of $f(z)$,

near z_0

then $$f(z) = \underbrace{\frac{b_n}{(z-z_0)^n} + \frac{b_{n-1}}{(z-z_0)^{n-1}} + \cdots + \frac{b_1}{z-z_0}} + \underline{\text{regular terms}}.$$

$h(z_0) \neq 0$.

$$f(z) = \frac{h(z)}{\underline{(z-z_0)^n}},$$ then Taylor expand $h(z)$ to get the above expression.