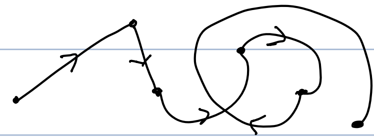
- Homotopy between 2 curves
  - what is a curve in Ω ? where $\Omega \subset \mathbb{C}$ is a open connected subset ?

$\gamma : [a,b] \to \Omega$, piecewise ($C^1$)-smooth.

i.e. $\exists \; a = t_0 < t_1 < \cdots < t_n = b$, $\gamma : [t_i, t_{i+1}] \to \Omega$ is a continuously differentiable function.

ex :

γ :



- homotopy between 2 curves is "an interpolation between two curves"

$\gamma_0, \gamma_1 : [a,b] \to \Omega$ curves. $\gamma_0(a) = \gamma_1(a) = \alpha$, $\gamma_0(b) = \gamma_1(b) = \beta$.

homotopy : $\gamma_s : [a,b] \to \Omega$ $\quad s \in [0,1]$.

s.t. ① $\gamma_s(t)\big|_{s=0} = \gamma_0(t)$
$\gamma_s(t)\big|_{s=1} = \gamma_1(t)$ $\Big\}\ \forall\, t \in [a,b]$.

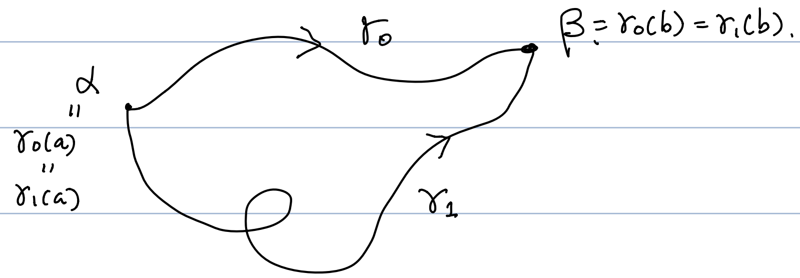
"for fixed s" ② $\forall\, s \in [0,1]$, $\gamma_s : [a,b] \to \Omega$ a curve.,
$\gamma_s(a) = \alpha$, $\gamma_s(b) = \beta$.

③ we want the function, $\gamma(s,t) = \gamma_s(t)$

$\gamma : [0,1] \times [a,b] \to \Omega$

to be jointly continuous in s and t.

(the family of curves varies continuously in s).



• If γ_0 and γ_1 are two curves in Ω with the same end points, and if γ_0, γ_1 are connected by a homotopy γ_s, then we say <u>γ_0 and γ_1 homotopic</u>.

<u>Thm</u>: Let Ω be an open connected set.

$f: \Omega \to \mathbb{C}$ hol'c.

γ_0, γ_1 homotopic curves in Ω,

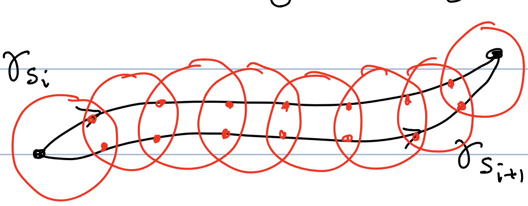
Then
$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz.$$

("deformation invariance of contour integral")

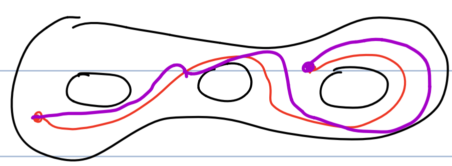
<u>Pf sketch</u>: • cut the s-interval $[0,1]$ into small enough segments. $0 < s_0 < s_1 < \cdots < s_N = 1$.

so that γ_{s_i}, $\gamma_{s_{i+1}}$ are "close together"

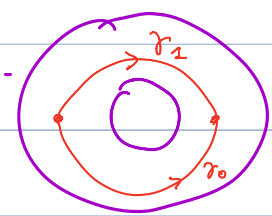
• Do a direct computation for comparing integral along such ~~cto~~ nearby curves. by covering them with disks.



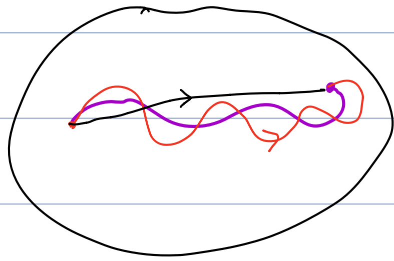
Ex:



Ω: domain with 3 holes.



γ_0, γ_1 are not homotopic.
since there is a hole in Ω.



<u>claim</u>: let $\Omega = \mathbb{D}$ a disk.

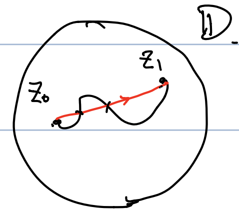
$z_0, z_1 \in \Omega$ be two distinct points.

then any curve from z_0 to z_1 is homotopic to the straight segments.

$\gamma_0 : [0,1] \longrightarrow \mathbb{D}$

$\gamma_1 : [0,1] \longrightarrow \mathbb{D}$

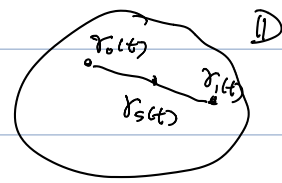
} any two curves ending at z_0, z_1.



· $\gamma_s(t) = (1-s) \cdot \gamma_0(t) + s \cdot \gamma_1(t)$

for each fixed t. $\gamma_s(t)$ lies on the segment between $\gamma_0(t)$ and $\gamma_1(t)$., that segment lies in $\mathbb{D}$.

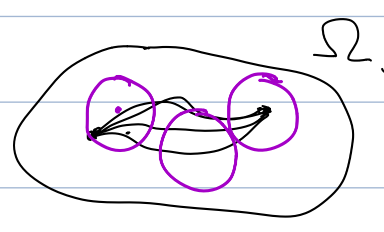
($\because$ $\mathbb{D}$ is convex)



$\gamma_s(t)$ is indeed a curve in $\mathbb{D}$, ending at z_0, and z_1.

① Choose ε small enough, such that $\forall s \in [0,1]$, $\forall t \in [a,b]$,

$$D_{3\varepsilon}(\gamma_s(t)) \subset \Omega.$$



· let K = the image of the map

$\gamma : [0,1] \times [a,b] \to \Omega$.

γ is continuous, $[0,1]\times[a,b]$ is compact

$\Rightarrow$ K is compact.

$\Rightarrow$ $\underline{\text{dist}(K, \Omega^c) > 0}$. $\quad$ (if $\Omega^c \neq \phi$)

Pf: $\cdot$ $\text{dist}(K,\Omega^c) = \inf_{x\in K} \text{dist}(x,\Omega^c) = \inf_{x\in K}\inf_{y\in\Omega^c} d(x,y)$.

$\cdot$ suppose $\text{dist}(K,\Omega^c) \underline{=0}$, then $\exists$ $(x_1,y_1), (x_2,y_2), \cdots$

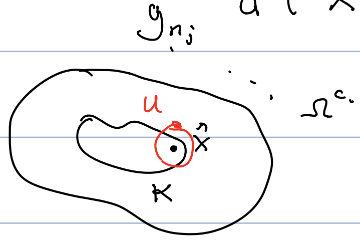
$x_i\in K$, $y_i\in\Omega^c$, s.t. $d(x_i,y_i) < \frac{1}{i}$, by compactness of K,

$\exists$ subsequence of $\{x_n\}$, called $x_{n_1}, x_{n_2}, \cdots$, s.t.

$\lim_{j\to\infty} x_{n_j} = \hat{x} \in K$. Then we get

$$d(\underline{x_{n_j}}, y_{n_j}) < \frac{1}{n_j} \to 0 \quad \text{as } j\to\infty$$

$$d(\hat{x}, y_{n_j}) \to 0 \quad \text{as } j\to\infty$$



$\hat{x}\in\Omega$, $\exists$ $U \ni x$, $U\subset\Omega$.

we have a contradiction:

if $d(\hat{x}, y_{n_j}) \to 0$, y_{n_j} will eventually be in U,

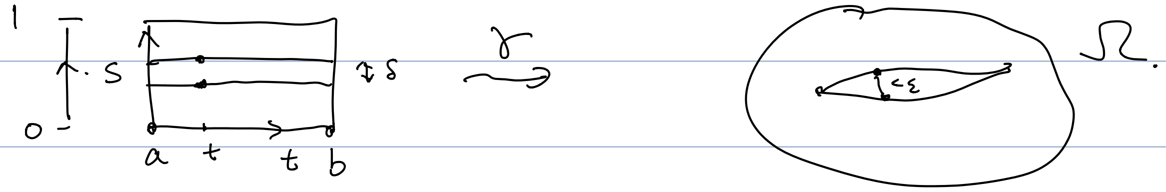
contradicting of $y_{n_j} \notin \Omega$.

$\cdot$ $3\varepsilon = \frac{1}{2}\cdot\text{dist}(K,\Omega^c)$.

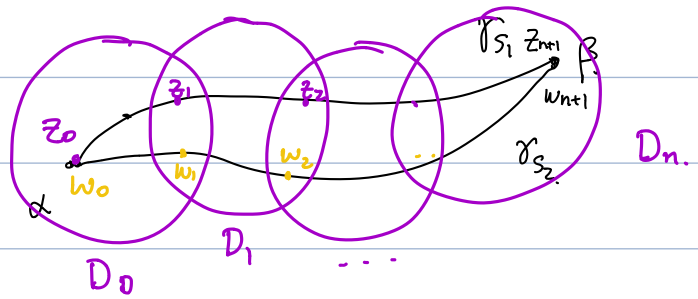
② By uniform continuity of $\gamma: [0,1]\times[a,b]\to\Omega$.

$\exists$ δ, such that $\forall$ $|s_1-s_2|<\delta$, we have

$$\sup_{t\in[a,b]} |\gamma_{s_1}(t) - \gamma_{s_2}(t)| < \varepsilon.$$



③. For any pair of s_1, s_2, s.t. $|s_1 - s_2| < \delta$, we will show $\int_{\gamma_{s_1}} f(z)\,dz = \int_{\gamma_{s_2}} f(z)\,dz$.



$z_0 = w_0 = \alpha$

$z_{n+1} = w_{n+1} = \beta$.

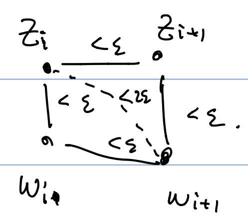
- cover both curves by disks $D_0, \cdots, D_n$ of <u>radius 2ε</u>, such that. $\exists$ $z_0, z_1, \cdots, z_{n+1}$ on γ_{s_1} and $w_0, w_1, \cdots, w_{n+1}$ on γ_{s_2},

  D_m contains. z_m, z_{m+1} / <u>w_m, w_{m+1}.</u> $\qquad D_1$: z_1, z_2 / w_1, w_2

Example construction: we choose some $t_0 < t_1 < t_2 \cdots$

s.t. $\left|\gamma_{s_1}(t_i) - \gamma_{s_1}(t_{i+1})\right| < \varepsilon$ (where $\gamma_{s_1}(t_i) = z_i$, $\gamma_{s_1}(t_{i+1}) = z_{i+1}$)

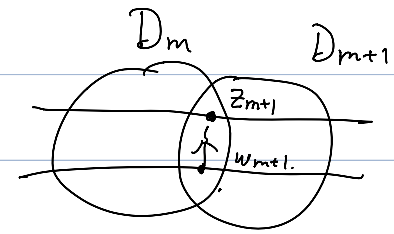
$\left|\gamma_{s_2}(t_i) - \gamma_{s_2}(t_{i+1})\right| < \varepsilon$ (where $\gamma_{s_2}(t_i) = w_i$, $\gamma_{s_2}(t_{i+1}) = w_{i+1}$)



one can then choose $D_i = D_{2\varepsilon}(z_i)$.

- f is hol'c on D_m, $\exists$ primitive F_m of f on D_m.
- on the over lap $D_m \cap D_{m+1}$, where both F_m and F_{m+1} are defined, they at most differ by a constant.

$$F_{m+1}(z) - F_m(z) = c \quad \text{on} \quad D_m \cap D_{m+1}.$$



$$F_{m+1}(z_{m+1}) - F_m(z_{m+1}) = c$$
$$F_{m+1}(w_{m+1}) - F_m(w_{m+1}) = c.$$

subtract them

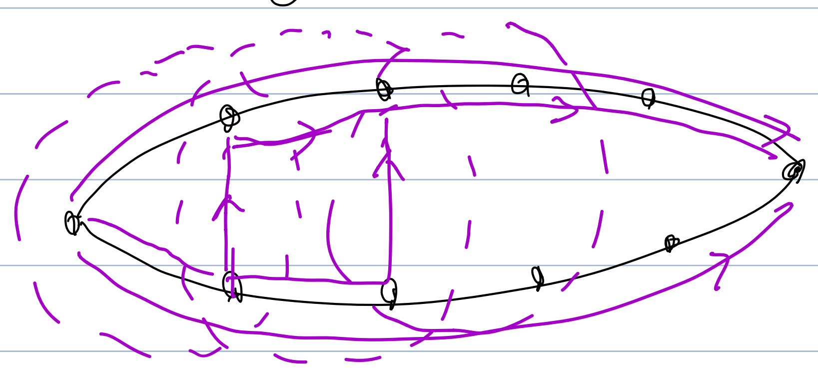
$$F_{m+1}(z_{m+1}) - F_{m+1}(w_{m+1}) = F_m(z_{m+1}) - F_m(w_{m+1}).$$
$$= \int_{w_{m+1}}^{z_{m+1}} f\, dz$$

① $$\int_{\gamma_{s_1}} f\, dz = \int_{z_0}^{z_1} f\, dz + \int_{z_1}^{z_2} f\, dz + \cdots + \int_{z_n}^{z_{n+1}} f\, dz.$$

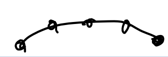
$$= F_0(z_1) - F_0(z_0) + F_1(z_2) - F_1(z_1) + \cdots + F_n(z_{n+1}) - F_n(z_n)$$

② $$\int_{\gamma_{s_2}} f\, dz = F_0(w_1) - F_0(w_0) + F_1(w_2) - F_1(w_1) + \cdots + F_n(w_{n+1}) - F_n(w_n).$$

① - ② $$= -F_0(z_0) - (-F_0(w_0)) + F_n(z_{n+1}) - F_n(w_{n+1}).$$
$$= 0.$$

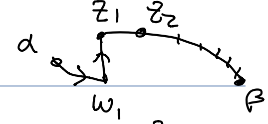


by doing local modification within disks, we can

turn a curve $\alpha, z_1, z_2, \cdots, z_n, \beta$ 

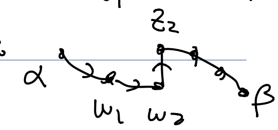
$\downarrow$

$\alpha, w_1, z_1, z_2, \dots, z_n, \beta$



$\downarrow$

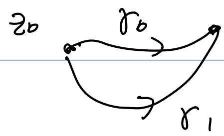
$\alpha, w_1, w_2, z_2, z_3, \dots, z_n, \beta.$



(region = connected, open),

$\#$.

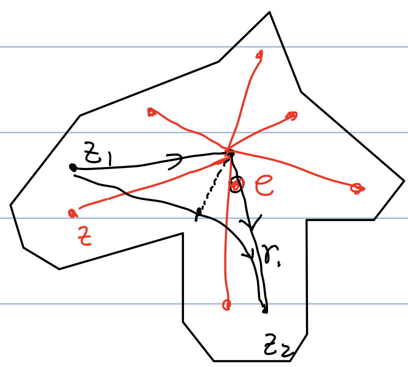
• **Def**: simply connected region, " for any 2 curves with the same endpoints are homotopic.



Ex: • convex region are simply connected.

(see the previous discussion on linear interpolation).

• "star-shaped domain", $\exists$ an origin $e \in \Omega$ s.t. any point $z \in \Omega$ can be connected to e by a straight segment



• interior of "toy contour"

• failure of simply connected: if Ω has "holes", "puncture",

Thm: If Ω is a simply connected ~~dom~~ region

$f: \Omega \to \mathbb{C}$ hol'c

then f has a primitive F.

Pf: • pick a $z_0 \in \Omega$, then define

$$F(z) = \int_{z_0}^{z} f(w)\, dw.$$

along any path from z_0 to z.

thm 5.1

this is well defined, since the integral only depends on the homotopy class of paths from z_0 to z_1 and there is a unique homotopy class of path from z_0 to z_1.

definition of simply connect domain

• check $F'(z) = f(z)$. ...

#.

• Conservative Vector field.

(2d): $V_x(x,y)$, $V_y(x,y)$, such that

$$\partial_y V_x = \partial_x V_y$$

$$v = V_x \cdot dx + V_y \cdot dy.$$

$\int_\gamma v$ only depends on the homotopy class.

• Q: $f: \Omega \to \mathbb{C}$ hol'c.

γ closed curve in Ω.

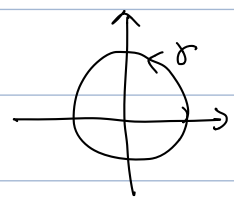
$w = f(z)$

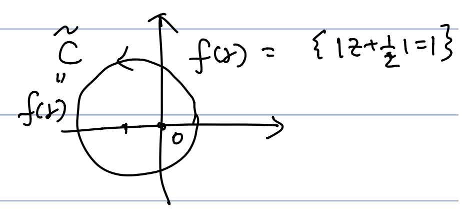
Q: $\displaystyle\int_\gamma \frac{f'(z)}{f(z)} \cdot dz = \int_{f(\gamma)} \frac{1}{w}\, dw.$

Ex: $f(z) = z - \frac{1}{z}$. γ = unit circle.

$$\underbrace{\int_\gamma \frac{f'(z)}{f(z)} \cdot dz}_{} = \int_{|z|=1} \frac{1}{z-\frac{1}{2}}\, dz = 2\pi i.$$

$$\int_{f(\gamma)} \frac{1}{w}\, dw = 2\pi i.$$



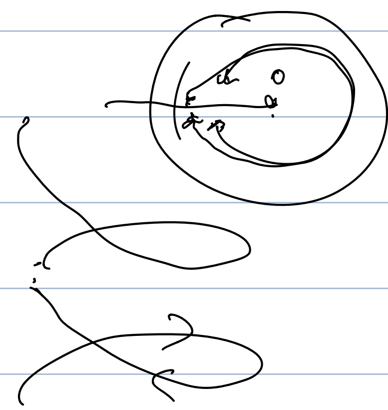


$$\frac{1}{w} \cdot dw = d \cdot \ln(w)$$

$$\underline{\ln(w)} = \underbrace{\ln \cdot |w|}_{\uparrow \text{ is well-defined}} + \underbrace{i \cdot \arg(w)}_{\uparrow \text{ is } \underline{\text{multivalued}}.}$$

$$\int_C d \cdot \ln|w| = 0$$

$$\int d \cdot (i \cdot \arg w) = i \cdot 2\pi.$$



HW6 #2.(2)

"Laurent expansion around a pole"

$f(z)$ is meromorphic, z_0 is a pole of order n,

Laurent expansion: $$f(z) = \underbrace{\frac{b_n}{(z-z_0)^n} + \frac{b_{n-1}}{(z-z_0)^{n-1}} + \cdots + \frac{b_1}{z-z_0}}_{} + \underbrace{a_0 + a_1(z-z_0) + \cdots}_{\text{regular}}$$

part

$\frac{h(z)}{(z-z_0)^n}$, then Taylor expand $h(z)$ around z_0

$$h(z) = \underline{c_0} + \underline{c_1}(z-z_0) + \underline{c_2}(z-z_0)^2 + \cdots$$