and since the sum converges uniformly for $z \in K$, the approximation by partial sums proves our claim.

This result allows us to travel from z_0 to z_1 through the finite sequence $\{w_j\}$ to find that $1/(z - z_0)$ can be approximated uniformly on K by polynomials in $1/(z - z_1)$. This concludes the proof of the lemma, and also that of the theorem.

6 Exercises

1. Prove that

$$\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$$

These are the **Fresnel integrals**. Here, $\int_0^\infty$ is interpreted as $\lim_{R\to\infty}\int_0^R$.

[Hint: Integrate the function e^{-z^2} over the path in Figure 14. Recall that $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$.]

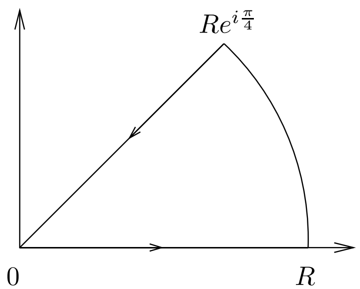

Figure 14. The contour in Exercise 1

2. Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

[Hint: The integral equals $\frac{1}{2i}\int_{-\infty}^\infty \frac{e^{ix}-1}{x}\,dx$. Use the indented semicircle.]

3. Evaluate the integrals

$$\int_0^\infty e^{-ax}\cos bx\,dx \quad \text{and} \quad \int_0^\infty e^{-ax}\sin bx\,dx\,, \qquad a > 0$$

by integrating e^{-Az}, $A = \sqrt{a^2+b^2}$, over an appropriate sector with angle ω, with $\cos\omega = a/A$.

4. Prove that for all $\xi \in \mathbb{C}$ we have $e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\xi}\, dx$.

5. Suppose f is continuously *complex* differentiable on Ω, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω. Apply Green's theorem to show that

$$\int_T f(z)\, dz = 0.$$

This provides a proof of Goursat's theorem under the additional assumption that f' is continuous.

[Hint: Green's theorem says that if (F, G) is a continuously differentiable vector field, then

$$\int_T F\, dx + G\, dy = \int_{\text{Interior of } T} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dxdy.$$

For appropriate F and G, one can then use the Cauchy-Riemann equations.]

6. Let Ω be an open subset of $\mathbb{C}$ and let $T \subset \Omega$ be a triangle whose interior is also contained in Ω. Suppose that f is a function holomorphic in Ω except possibly at a point w inside T. Prove that if f is bounded near w, then

$$\int_T f(z)\, dz = 0.$$

7. Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z,\, w \in \mathbb{D}} |f(z) - f(w)|$ of the image of f satisfies

$$2|f'(0)| \leq d.$$

Moreover, it can be shown that equality holds precisely when f is linear, $f(z) = a_0 + a_1 z$.

Note. In connection with this result, see the relationship between the diameter of a curve and Fourier series described in Problem 1, Chapter 4, Book I.

[Hint: $2f'(0) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta) - f(-\zeta)}{\zeta^2}\, d\zeta$ whenever $0 < r < 1$.]

8. If f is a holomorphic function on the strip $-1 < y < 1$, $x \in \mathbb{R}$ with

$$|f(z)| \leq A(1 + |z|)^\eta, \qquad \eta \text{ a fixed real number}$$

for all z in that strip, show that for each integer $n \geq 0$ there exists $A_n \geq 0$ so that

$$|f^{(n)}(x)| \leq A_n(1 + |x|)^\eta, \qquad \text{for all } x \in \mathbb{R}.$$

[Hint: Use the Cauchy inequalities.]

9. Let Ω be a bounded open subset of $\mathbb{C}$, and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that

$$\varphi(z_0) = z_0 \quad \text{and} \quad \varphi'(z_0) = 1$$

then φ is linear.
[Hint: Why can one assume that $z_0 = 0$? Write $\varphi(z) = z + a_n z^n + O(z^{n+1})$ near 0, and prove that if $\varphi_k = \varphi \circ \cdots \circ \varphi$ (where φ appears k times), then $\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$. Apply the Cauchy inequalities and let $k \to \infty$ to conclude the proof. Here we use the standard O notation, where $f(z) = O(g(z))$ as $z \to 0$ means that $|f(z)| \leq C|g(z)|$ for some constant C as $|z| \to 0$.]

10. Weierstrass's theorem states that a continuous function on $[0, 1]$ can be uniformly approximated by polynomials. Can every continuous function on the closed unit disc be approximated uniformly by polynomials in the variable z?

11. Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0.

(a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi}) \operatorname{Re}\left(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z} \right) d\varphi.$$

(b) Show that

$$\operatorname{Re}\left(\frac{Re^{i\gamma} + r}{Re^{i\gamma} - r} \right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\gamma + r^2}.$$

[Hint: For the first part, note that if $w = R^2/\overline{z}$, then the integral of $f(\zeta)/(\zeta - w)$ around the circle of radius R centered at the origin is zero. Use this, together with the usual Cauchy integral formula, to deduce the desired identity.]

12. Let u be a real-valued function defined on the unit disc $\mathbb{D}$. Suppose that u is twice continuously differentiable and harmonic, that is,

$$\triangle u(x, y) = 0$$

for all $(x, y) \in \mathbb{D}$.

(a) Prove that there exists a holomorphic function f on the unit disc such that

$$\operatorname{Re}(f) = u.$$

Also show that the imaginary part of f is uniquely defined up to an additive (real) constant. [Hint: From the previous chapter we would have $f'(z) = 2\partial u/\partial z$. Therefore, let $g(z) = 2\partial u/\partial z$ and prove that g is holomorphic. Why can one find F with $F' = g$? Prove that $\operatorname{Re}(F)$ differs from u by a real constant.]

(b) Deduce from this result, and from Exercise 11, the Poisson integral representation formula from the Cauchy integral formula: If u is harmonic in the unit disc and continuous on its closure, then if $z = re^{i\theta}$ one has

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - \varphi)u(\varphi)\, d\varphi$$

where $P_r(\gamma)$ is the Poisson kernel for the unit disc given by

$$P_r(\gamma) = \frac{1 - r^2}{1 - 2r\cos\gamma + r^2}.$$

13. Suppose f is an analytic function defined everywhere in $\mathbb{C}$ and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.

[Hint: Use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.]

14. Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f in the open unit disc, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

15. Suppose f is a non-vanishing continuous function on $\overline{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$. Prove that if

$$|f(z)| = 1 \qquad \text{whenever } |z| = 1,$$

then f is constant.

[Hint: Extend f to all of $\mathbb{C}$ by $f(z) = 1/\overline{f(1/\overline{z})}$ whenever $|z| > 1$, and argue as in the Schwarz reflection principle.]

7 Problems

1. Here are some examples of analytic functions on the unit disc that cannot be extended analytically past the unit circle. The following definition is needed. Let

f be a function defined in the unit disc $\mathbb{D}$, with boundary circle C. A point w on C is said to be *regular* for f if there is an open neighborhood U of w and an analytic function g on U, so that $f = g$ on $\mathbb{D} \cap U$. A function f defined on $\mathbb{D}$ cannot be continued analytically past the unit circle if no point of C is regular for f.

(a) Let

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \qquad \text{for } |z| < 1.$$

Notice that the radius of convergence of the above series is 1. Show that f cannot be continued analytically past the unit disc. [Hint: Suppose $\theta = 2\pi p/2^k$, where p and k are positive integers. Let $z = re^{i\theta}$; then $|f(re^{i\theta})| \to \infty$ as $r \to 1$.]

(b) * Fix $0 < \alpha < \infty$. Show that the analytic function f defined by

$$f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n} \qquad \text{for } |z| < 1$$

extends continuously to the unit circle, but cannot be analytically continued past the unit circle. [Hint: There is a nowhere differentiable function lurking in the background. See Chapter 4 in Book I.]

2.* Let

$$F(z) = \sum_{n=1}^{\infty} d(n) z^n \qquad \text{for } |z| < 1$$

where $d(n)$ denotes the number of divisors of n. Observe that the radius of convergence of this series is 1. Verify the identity

$$\sum_{n=1}^{\infty} d(n) z^n = \sum_{n=1}^{\infty} \frac{z^n}{1 - z^n}.$$

Using this identity, show that if $z = r$ with $0 < r < 1$, then

$$|F(r)| \geq c \frac{1}{1-r} \log(1/(1-r))$$

as $r \to 1$. Similarly, if $\theta = 2\pi p/q$ where p and q are positive integers and $z = re^{i\theta}$, then

$$|F(re^{i\theta})| \geq c_{p/q} \frac{1}{1-r} \log(1/(1-r))$$

as $r \to 1$. Conclude that F cannot be continued analytically past the unit disc.

3. Morera's theorem states that if f is continuous in $\mathbb{C}$, and $\int_T f(z)\, dz = 0$ for all triangles T, then f is holomorphic in $\mathbb{C}$. Naturally, we may ask if the conclusion still holds if we replace triangles by other sets.

(a) Suppose that f is continuous on $\mathbb{C}$, and

$$\int_C f(z)\,dz = 0 \tag{16}$$

for every circle C. Prove that f is holomorphic.

(b) More generally, let Γ be any toy contour, and $\mathcal{F}$ the collection of all translates and dilates of Γ. Show that if f is continuous on $\mathbb{C}$, and

$$\int_\gamma f(z)\,dz = 0 \qquad \text{for all } \gamma \in \mathcal{F}$$

then f is holomorphic. In particular, Morera's theorem holds under the weaker assumption that $\int_T f(z)\,dz = 0$ for all equilateral triangles.

[Hint: As a first step, assume that f is twice real differentiable, and write $f(z) = f(z_0) + a(z - z_0) + b(\overline{z - z_0}) + O(|z - z_0|^2)$ for z near z_0. Integrating this expansion over small circles around z_0 yields $\partial f/\partial \overline{z} = b = 0$ at z_0. Alternatively, suppose only that f is differentiable and apply Green's theorem to conclude that the real and imaginary parts of f satisfy the Cauchy-Riemann equations.

In general, let $\varphi(w) = \varphi(x, y)$ (when $w = x + iy$) denote a smooth function with $0 \leq \varphi(w) \leq 1$, and $\int_{\mathbb{R}^2} \varphi(w)\,dV(w) = 1$, where $dV(w) = dxdy$, and $\int$ denotes the usual integral of a function of two variables in $\mathbb{R}^2$. For each $\epsilon > 0$, let $\varphi_\epsilon(z) = \epsilon^{-2}\varphi(\epsilon^{-1}z)$, as well as

$$f_\epsilon(z) = \int_{\mathbb{R}^2} f(z - w)\varphi_\epsilon(w)\,dV(w),$$

where the integral denotes the usual integral of functions of two variables, with $dV(w)$ the area element of $\mathbb{R}^2$. Then f_ϵ is smooth, satisfies condition (16), and $f_\epsilon \to f$ uniformly on any compact subset of $\mathbb{C}$.]

4. Prove the converse to Runge's theorem: if K is a compact set whose complement if not connected, then there exists a function f holomorphic in a neighborhood of K which cannot be approximated uniformly by polynomial on K.

[Hint: Pick a point z_0 in a bounded component of K^c, and let $f(z) = 1/(z - z_0)$. If f can be approximated uniformly by polynomials on K, show that there exists a polynomial p such that $|(z - z_0)p(z) - 1| < 1$. Use the maximum modulus principle (Chapter 3) to show that this inequality continues to hold for all z in the component of K^c that contains z_0.]

5.* There exists an entire function F with the following "universal" property: given any entire function h, there is an increasing sequence $\{N_k\}_{k=1}^\infty$ of positive integers, so that

$$\lim_{n\to\infty} F(z + N_k) = h(z)$$

uniformly on every compact subset of $\mathbb{C}$.

(a) Let $p_1, p_2, \ldots$ denote an enumeration of the collection of polynomials whose coefficients have rational real and imaginary parts. Show that it suffices to find an entire function F and an increasing sequence $\{M_n\}$ of positive integers, such that

$$|F(z) - p_n(z - M_n)| < \frac{1}{n} \quad \text{whenever } z \in D_n, \tag{17}$$

where D_n denotes the disc centered at M_n and of radius n. [Hint: Given h entire, there exists a sequence $\{n_k\}$ such that $\lim_{k\to\infty} p_{n_k}(z) = h(z)$ uniformly on every compact subset of $\mathbb{C}$.]

(b) Construct F satisfying (17) as an infinite series

$$F(z) = \sum_{n=1}^{\infty} u_n(z)$$

where $u_n(z) = p_n(z - M_n)e^{-c_n(z-M_n)^2}$, and the quantities $c_n > 0$ and $M_n > 0$ are chosen appropriately with $c_n \to 0$ and $M_n \to \infty$. [Hint: The function e^{-z^2} vanishes rapidly as $|z| \to \infty$ in the sectors $\{|\arg z| < \pi/4 - \delta\}$ and $\{|\pi - \arg z| < \pi/4 - \delta\}$.]

In the same spirit, there exists an alternate "universal" entire function G with the following property: given any entire function h, there is an increasing sequence $\{N_k\}_{k=1}^{\infty}$ of positive integers, so that

$$\lim_{k\to\infty} D^{N_k} G(z) = h(z)$$

uniformly on every compact subset of $\mathbb{C}$. Here $D^j G$ denotes the j^{th} (complex) derivative of G.