Corollary 7.3 *If f is holomorphic in a disc $D_R(z_0)$, and $u = \mathrm{Re}(f)$, then*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta, \qquad \text{for any } 0 < r < R.$$

Recall that u is harmonic whenever f is holomorphic, and in fact, the above corollary is a property enjoyed by every harmonic function in the disc $D_R(z_0)$. This follows from Exercise 12 in Chapter 2, which shows that every harmonic function in a disc is the real part of a holomorphic function in that disc.

8 Exercises

1. Using Euler's formula

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i},$$

show that the complex zeros of $\sin \pi z$ are exactly at the integers, and that they are each of order 1.

Calculate the residue of $1/\sin \pi z$ at $z = n \in \mathbb{Z}$.

2. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

Where are the poles of $1/(1+z^4)$?

3. Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\, dx = \pi\frac{e^{-a}}{a}, \qquad \text{for } a > 0.$$

4. Show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2}\, dx = \pi e^{-a}, \qquad \text{for all } a > 0.$$

5. Use contour integration to show that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{(1+x^2)^2}\, dx = \frac{\pi}{2}(1+2\pi|\xi|)e^{-2\pi|\xi|}$$

for all ξ real.

6. Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi.$$

7. Prove that

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}}, \qquad \text{whenever } a > 1.$$

8. Prove that

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

if $a > |b|$ and $a, b \in \mathbb{R}$.

9. Show that

$$\int_0^1 \log(\sin \pi x)\, dx = -\log 2.$$

[Hint: Use the contour shown in Figure 9.]

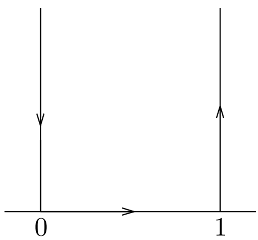

Figure 9. Contour in Exercise 9

10. Show that if $a > 0$, then

$$\int_0^{\infty} \frac{\log x}{x^2 + a^2}\, dx = \frac{\pi}{2a} \log a.$$

[Hint: Use the contour in Figure 10.]

11. Show that if $|a| < 1$, then

$$\int_0^{2\pi} \log|1 - ae^{i\theta}|\, d\theta = 0.$$

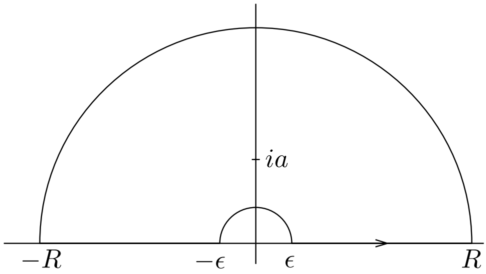

Figure 10. Contour in Exercise 10

Then, prove that the above result remains true if we assume only that $|a| \leq 1$.

12. Suppose u is not an integer. Prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin \pi u)^2}$$

by integrating

$$f(z) = \frac{\pi \cot \pi z}{(u+z)^2}$$

over the circle $|z| = R_N = N + 1/2$ (N integral, $N \geq |u|$), adding the residues of f inside the circle, and letting N tend to infinity.
Note. Two other derivations of this identity, using Fourier series, were given in Book I.

13. Suppose $f(z)$ is holomorphic in a punctured disc $D_r(z_0) - \{z_0\}$. Suppose also that

$$|f(z)| \leq A|z - z_0|^{-1+\epsilon}$$

for some $\epsilon > 0$, and all z near z_0. Show that the singularity of f at z_0 is removable.

14. Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$, and $a \neq 0$.

[Hint: Apply the Casorati-Weierstrass theorem to $f(1/z)$.]

15. Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

(a) Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \leq AR^k + B$$

for all $R > 0$, and for some integer $k \geq 0$ and some constants $A, B > 0$, then f is a polynomial of degree $\leq k$.

(b) Show that if f is holomorphic in the unit disc, is bounded, and converges uniformly to zero in the sector $\theta < \arg z < \varphi$ as $|z| \to 1$, then $f = 0$.

(c) Let $w_1, \ldots, w_n$ be points on the unit circle in the complex plane. Prove that there exists a point z on the unit circle such that the product of the distances from z to the points w_j, $1 \leq j \leq n$, is at least 1. Conclude that there exists a point w on the unit circle such that the product of the distances from w to the points w_j, $1 \leq j \leq n$, is exactly equal to 1.

(d) Show that if the real part of an entire function f is bounded, then f is constant.

16. Suppose f and g are holomorphic in a region containing the disc $|z| \leq 1$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $|z| \leq 1$. Let

$$f_\epsilon(z) = f(z) + \epsilon g(z).$$

Show that if ϵ is sufficiently small, then

(a) $f_\epsilon(z)$ has a unique zero in $|z| \leq 1$, and

(b) if z_ϵ is this zero, the mapping $\epsilon \mapsto z_\epsilon$ is continuous.

17. Let f be non-constant and holomorphic in an open set containing the closed unit disc.

(a) Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of f contains the unit disc. [Hint: One must show that $f(z) = w_0$ has a root for every $w_0 \in \mathbb{D}$. To do this, it suffices to show that $f(z) = 0$ has a root (why?). Use the maximum modulus principle to conclude.]

(b) If $|f(z)| \geq 1$ whenever $|z| = 1$ and there exists a point $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of f contains the unit disc.

18. Give another proof of the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

using homotopy of curves.

[Hint: Deform the circle C to a small circle centered at z, and note that the quotient $(f(\zeta) - f(z))/(\zeta - z)$ is bounded.]

19. Prove the maximum principle for harmonic functions, that is:

(a) If u is a non-constant real-valued harmonic function in a region Ω, then u cannot attain a maximum (or a minimum) in Ω.

(b) Suppose that Ω is a region with compact closure $\overline{\Omega}$. If u is harmonic in Ω and continuous in $\overline{\Omega}$, then

$$\sup_{z\in\Omega} |u(z)| \leq \sup_{z\in\overline{\Omega}-\Omega} |u(z)|.$$

[Hint: To prove the first part, assume that u attains a local maximum at z_0. Let f be holomorphic near z_0 with $u = \mathrm{Re}(f)$, and show that f is not open. The second part follows directly from the first.]

20. This exercise shows how the mean square convergence dominates the uniform convergence of analytic functions. If U is an open subset of $\mathbb{C}$ we use the notation

$$\|f\|_{L^2(U)} = \left(\int_U |f(z)|^2 \, dxdy\right)^{1/2}$$

for the mean square norm, and

$$\|f\|_{L^\infty(U)} = \sup_{z\in U} |f(z)|$$

for the sup norm.

(a) If f is holomorphic in a neighborhood of the disc $D_r(z_0)$, show that for any $0 < s < r$ there exists a constant $C > 0$ (which depends on s and r) such that

$$\|f\|_{L^\infty(D_s(z_0))} \leq C\|f\|_{L^2(D_r(z_0))}\,.$$

(b) Prove that if $\{f_n\}$ is a Cauchy sequence of holomorphic functions in the mean square norm $\|\cdot\|_{L^2(U)}$, then the sequence $\{f_n\}$ converges uniformly on every compact subset of U to a holomorphic function.

[Hint: Use the mean-value property.]

21. Certain sets have geometric properties that guarantee they are simply connected.

(a) An open set $\Omega \subset \mathbb{C}$ is **convex** if for any two points in Ω, the straight line segment between them is contained in Ω. Prove that a convex open set is simply connected.

(b) More generally, an open set $\Omega \subset \mathbb{C}$ is **star-shaped** if there exists a point $z_0 \in \Omega$ such that for any $z \in \Omega$, the straight line segment between z and z_0 is contained in Ω. Prove that a star-shaped open set is simply connected. Conclude that the slit plane $\mathbb{C} - \{(-\infty, 0]\}$ (and more generally any sector, convex or not) is simply connected.

(c) What are other examples of open sets that are simply connected?

22. Show that there is no holomorphic function f in the unit disc $\mathbb{D}$ that extends continuously to $\partial\mathbb{D}$ such that $f(z) = 1/z$ for $z \in \partial\mathbb{D}$.

9 Problems

1.* Consider a holomorphic map on the unit disc $f : \mathbb{D} \to \mathbb{C}$ which satisfies $f(0) = 0$. By the open mapping theorem, the image $f(\mathbb{D})$ contains a small disc centered at the origin. We then ask: does there exist $r > 0$ such that for *all* $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 0$, we have $D_r(0) \subset f(\mathbb{D})$?

(a) Show that with no further restrictions on f, no such r exists. It suffices to find a sequence of functions $\{f_n\}$ holomorphic in $\mathbb{D}$ such that $1/n \notin f(\mathbb{D})$. Compute $f_n'(0)$, and discuss.

(b) Assume in addition that f also satisfies $f'(0) = 1$. Show that despite this new assumption, there exists no $r > 0$ satisfying the desired condition.

[Hint: Try $f_\epsilon(z) = \epsilon(e^{z/\epsilon} - 1)$.]

The Koebe-Bieberbach theorem states that if in addition to $f(0) = 0$ and $f'(0) = 1$ we also assume that f is injective, then such an r exists and the best possible value is $r = 1/4$.

(c) As a first step, show that if $h(z) = \frac{1}{z} + c_0 + c_1 z + c_2 z^2 + \cdots$ is analytic and injective for $0 < |z| < 1$, then $\sum_{n=1}^{\infty} n|c_n|^2 \leq 1$.

[Hint: Calculate the area of the complement of $h(D_\rho(0) - \{0\})$ where $0 < \rho < 1$, and let $\rho \to 1$.]

(d) If $f(z) = z + a_2 z^2 + \cdots$ satisfies the hypotheses of the theorem, show that there exists another function g satisfying the hypotheses of the theorem such that $g^2(z) = f(z^2)$.

[Hint: $f(z)/z$ is nowhere vanishing so there exists ψ such that $\psi^2(z) = f(z)/z$ and $\psi(0) = 1$. Check that $g(z) = z\psi(z^2)$ is injective.]

(e) With the notation of the previous part, show that $|a_2| \leq 2$, and that equality holds if and only if

$$f(z) = \frac{z}{(1 - e^{i\theta} z)^2} \quad \text{for some } \theta \in \mathbb{R}.$$

[Hint: What is the power series expansion of $1/g(z)$? Use part (c).]

(f) If $h(z) = \frac{1}{z} + c_0 + c_1 z + c_2 z^2 + \cdots$ is injective on $\mathbb{D}$ and avoids the values z_1 and z_2, show that $|z_1 - z_2| \leq 4$.

[Hint: Look at the second coefficient in the power series expansion of $1/(h(z) - z_j)$.]

(g) Complete the proof of the theorem. [Hint: If f avoids w, then $1/f$ avoids 0 and $1/w$.]

2. Let u be a harmonic function in the unit disc that is continuous on its closure. Deduce Poisson's integral formula

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z_0|^2}{|e^{i\theta} - z_0|^2} u(e^{i\theta})\, d\theta \qquad \text{for } |z_0| < 1$$

from the special case $z_0 = 0$ (the mean value theorem). Show that if $z_0 = re^{i\varphi}$, then

$$\frac{1 - |z_0|^2}{|e^{i\theta} - z_0|^2} = \frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2} = P_r(\theta - \varphi),$$

and we recover the expression for the Poisson kernel derived in the exercises of the previous chapter.

[Hint: Set $u_0(z) = u(T(z))$ where

$$T(z) = \frac{z_0 - z}{1 - \overline{z_0} z}.$$

Prove that u_0 is harmonic. Then apply the mean value theorem to u_0, and make a change of variables in the integral.]

3. If $f(z)$ is holomorphic in the deleted neighborhood $\{0 < |z - z_0| < r\}$ and has a pole of order k at z_0, then we can write

$$f(z) = \frac{a_{-k}}{(z - z_0)^k} + \cdots + \frac{a_{-1}}{(z - z_0)} + g(z)$$

where g is holomorphic in the disc $\{|z - z_0| < r\}$. There is a generalization of this expansion that holds even if z_0 is an essential singularity. This is a special case of the **Laurent series expansion**, which is valid in an even more general setting.

Let f be holomorphic in a region containing the annulus $\{z : r_1 \leq |z - z_0| \leq r_2\}$ where $0 < r_1 < r_2$. Then,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where the series converges absolutely in the interior of the annulus. To prove this, it suffices to write

$$f(z) = \frac{1}{2\pi i} \int_{C_{r_2}} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{C_{r_1}} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

when $r_1 < |z - z_0| < r_2$, and argue as in the proof of Theorem 4.4, Chapter 2. Here C_{r_1} and C_{r_2} are the circles bounding the annulus.

4.* Suppose Ω is a bounded region. Let L be a (two-way infinite) line that intersects Ω. Assume that $\Omega \cap L$ is an interval I. Choosing an orientation for L, we can define Ω_l and Ω_r to be the subregions of Ω lying strictly to the left or right of L, with $\Omega = \Omega_l \cup I \cup \Omega_r$ a disjoint union. If Ω_l and Ω_r are simply connected, then Ω is simply connected.

5.* Let

$$g(z) = \frac{1}{2\pi i} \int_{-M}^{M} \frac{h(x)}{x - z}\, dx$$

where h is continuous and supported in $[-M, M]$.

(a) Prove that the function g is holomorphic in $\mathbb{C} - [-M, M]$, and vanishes at infinity, that is, $\lim_{|z|\to\infty} |g(z)| = 0$. Moreover, the "jump" of g across $[-M, M]$ is h, that is,

$$h(x) = \lim_{\epsilon\to 0, \epsilon > 0} g(x + i\epsilon) - g(x - i\epsilon).$$

[Hint: Express the difference $g(x + i\epsilon) - g(x - i\epsilon)$ in terms of a convolution of h with the Poisson kernel.]

(b) If h satisfies a mild smoothness condition, for instance a Hölder condition with exponent α, that is, $|h(x) - h(y)| \leq C|x - y|^\alpha$ for some $C > 0$ and all $x, y \in [-M, M]$, then $g(x + i\epsilon)$ and $g(x - i\epsilon)$ converge uniformly to functions $g_+(x)$ and $g_-(x)$ as $\epsilon \to 0$. Then, g can be characterized as the unique holomorphic function that satisfies:

(i) g is holomorphic outside $[-M, M]$,

(ii) g vanishes at infinity,

(iii) $g(x + i\epsilon)$ and $g(x - i\epsilon)$ converge uniformly as $\epsilon \to 0$ to functions $g_+(x)$ and $g_-(x)$ with

$$g_+(x) - g_-(x) = h(x).$$

[Hint: If G is another function satisfying these conditions, $g - G$ is entire.]