for all $\xi < 0$. Then, passing to the limit successively, one has $\widehat{f_{\epsilon,0}}(\xi) = 0$ for $\xi < 0$, and finally $\hat{f}(\xi) = \widehat{f_{0,0}}(\xi) = 0$ for all $\xi < 0$.

Remark. The reader should note a certain analogy between the above theorem and Theorem 7.1 in Chapter 3. Here we are dealing with a function holomorphic in the upper half-plane, and there with a function holomorphic in a disc. In the present case the Fourier transform vanishes when $\xi < 0$, and in the earlier case, the Fourier coefficients vanish when $n < 0$.

4 Exercises

1. Suppose f is continuous and of moderate decrease, and $\hat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}$. Show that $f = 0$ by completing the following outline:

(a) For each fixed real number t consider the two functions

$$A(z) = \int_{-\infty}^{t} f(x)e^{-2\pi iz(x-t)}\, dx \quad \text{and} \quad B(z) = -\int_{t}^{\infty} f(x)e^{-2\pi iz(x-t)}\, dx.$$

Show that $A(\xi) = B(\xi)$ for all $\xi \in \mathbb{R}$.

(b) Prove that the function F equal to A in the closed upper half-plane, and B in the lower half-plane, is entire and bounded, thus constant. In fact, show that $F = 0$.

(c) Deduce that

$$\int_{-\infty}^{t} f(x)\, dx = 0,$$

for all t, and conclude that $f = 0$.

2. If $f \in \mathfrak{F}_a$ with $a > 0$, then for any positive integer n one has $f^{(n)} \in \mathfrak{F}_b$ whenever $0 \leq b < a$.

[Hint: Modify the solution to Exercise 8 in Chapter 2.]

3. Show, by contour integration, that if $a > 0$ and $\xi \in \mathbb{R}$ then

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a}{a^2 + x^2} e^{-2\pi ix\xi}\, dx = e^{-2\pi a|\xi|},$$

and check that

$$\int_{-\infty}^{\infty} e^{-2\pi a|\xi|} e^{2\pi i\xi x}\, d\xi = \frac{1}{\pi} \frac{a}{a^2 + x^2}.$$

4. Suppose Q is a polynomial of degree ≥ 2 with distinct roots, none lying on the real axis. Calculate

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{Q(x)}\,dx, \qquad \xi \in \mathbb{R}$$

in terms of the roots of Q. What happens when several roots coincide?
[Hint: Consider separately the cases $\xi < 0$, $\xi = 0$, and $\xi > 0$. Use residues.]

5. More generally, let $R(x) = P(x)/Q(x)$ be a rational function with (degree Q) $\geq$ (degree$P$)+2 and $Q(x) \neq 0$ on the real axis.

(a) Prove that if $\alpha_1, \ldots, \alpha_k$ are the roots of R in the upper half-plane, then there exists polynomials $P_j(\xi)$ of degree less than the multiplicity of α_j so that

$$\int_{-\infty}^{\infty} R(x)e^{-2\pi i x\xi}\,dx = \sum_{j=1}^{k} P_j(\xi)e^{-2\pi i \alpha_j \xi}, \qquad \text{when } \xi < 0.$$

(b) In particular, if $Q(z)$ has no zeros in the upper half-plane, then $\int_{-\infty}^{\infty} R(x)e^{-2\pi i x\xi}\,dx = 0$ for $\xi < 0$.

(c) Show that similar results hold in the case $\xi > 0$.

(d) Show that

$$\int_{-\infty}^{\infty} R(x)e^{-2\pi i x\xi}\,dx = O(e^{-a|\xi|}), \qquad \xi \in \mathbb{R}$$

as $|\xi| \to \infty$ for some $a > 0$. Determine the best possible a's in terms of the roots of R.

[Hint: For part (a), use residues. The powers of ξ appear when one differentiates the function $f(z) = R(z)e^{-2\pi i z\xi}$ (as in the formula of Theorem 1.4 in the previous chapter). For part (c) argue in the lower half-plane.]

6. Prove that

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{a}{a^2+n^2} = \sum_{n=-\infty}^{\infty} e^{-2\pi a|n|}$$

whenever $a > 0$. Hence show that the sum equals $\coth \pi a$.

7. The Poisson summation formula applied to specific examples often provides interesting identities.

(a) Let τ be fixed with $\text{Im}(\tau) > 0$. Apply the Poisson summation formula to

$$f(z) = (\tau + z)^{-k},$$

where k is an integer ≥ 2, to obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} e^{2\pi i m \tau}.$$

(b) Set $k = 2$ in the above formula to show that if $\text{Im}(\tau) > 0$, then

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^2} = \frac{\pi^2}{\sin^2(\pi\tau)}.$$

(c) Can one conclude that the above formula holds true whenever τ is any complex number that is not an integer?

[Hint: For (a), use residues to prove that $\hat{f}(\xi) = 0$, if $\xi < 0$, and

$$\hat{f}(\xi) = \frac{(-2\pi i)^k}{(k-1)!} \xi^{k-1} e^{2\pi i \xi \tau}, \qquad \text{when } \xi > 0.]$$

8. Suppose $\hat{f}$ has compact support contained in $[-M, M]$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Show that

$$a_n = \frac{(2\pi i)^n}{n!} \int_{-M}^{M} \hat{f}(\xi) \xi^n \, d\xi,$$

and as a result

$$\limsup_{n\to\infty} (n!|a_n|)^{1/n} \leq 2\pi M.$$

In the converse direction, let f be *any* power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\limsup_{n\to\infty} (n!|a_n|)^{1/n} \leq 2\pi M$. Then, f is holomorphic in the complex plane, and for every $\epsilon > 0$ there exists $A_\epsilon > 0$ such that

$$|f(z)| \leq A_\epsilon e^{2\pi(M+\epsilon)|z|}.$$

9. Here are further results similar to the Phragmén-Lindelöf theorem.

(a) Let F be a holomorphic function in the right half-plane that extends continuously to the boundary, that is, the imaginary axis. Suppose that $|F(iy)| \leq 1$ for all $y \in \mathbb{R}$, and

$$|F(z)| \leq Ce^{c|z|^\gamma}$$

for some $c, C > 0$ and $\gamma < 1$. Prove that $|F(z)| \leq 1$ for all z in the right half-plane.

(b) More generally, let S be a sector whose vertex is the origin, and forming an angle of π/β. Let F be a holomorphic function in S that is continuous on the closure of S, so that $|F(z)| \leq 1$ on the boundary of S and

$$|F(z)| \leq Ce^{c|z|^\alpha} \text{ for all } z \in S$$

for some $c, C > 0$ and $0 < \alpha < \beta$. Prove that $|F(z)| \leq 1$ for all $z \in S$.

10. This exercise generalizes some of the properties of $e^{-\pi x^2}$ related to the fact that it is its own Fourier transform.

Suppose $f(z)$ is an entire function that satisfies

$$|f(x+iy)| \leq ce^{-ax^2+by^2}$$

for some $a, b, c > 0$. Let

$$\hat{f}(\zeta) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\zeta}\,dx.$$

Then, $\hat{f}$ is an entire function of ζ that satisfies

$$|\hat{f}(\xi + i\eta)| \leq c'e^{-a'\xi^2+b'\eta^2}$$

for some $a', b', c' > 0$.

[Hint: To prove $\hat{f}(\xi) = O(e^{-a'\xi^2})$, assume $\xi > 0$ and change the contour of integration to $x - iy$ for some $y > 0$ fixed, and $-\infty < x < \infty$. Then

$$\hat{f}(\xi) = O(e^{-2\pi y\xi}e^{by^2}).$$

Finally, choose $y = d\xi$ where d is a small constant.]

11. One can give a neater formulation of the result in Exercise 10 by proving the following fact.

Suppose $f(z)$ is an entire function of strict order 2, that is,

$$f(z) = O(e^{c_1|z|^2})$$

for some $c_1 > 0$. Suppose also that for x real,

$$f(x) = O(e^{-c_2|x|^2})$$

for some $c_2 > 0$. Then

$$|f(x+iy)| = O(e^{-ax^2+by^2})$$

for some $a, b > 0$. The converse is obviously true.

12. The principle that a function and its Fourier transform cannot both be too small at infinity is illustrated by the following theorem of Hardy.

If f is a function on $\mathbb{R}$ that satisfies

$$f(x) = O(e^{-\pi x^2}) \quad \text{and} \quad \hat{f}(\xi) = O(e^{-\pi \xi^2}),$$

then f is a constant multiple of $e^{-\pi x^2}$. As a result, if $f(x) = O(e^{-\pi A x^2})$, and $\hat{f}(\xi) = O(e^{-\pi B \xi^2})$, with $AB > 1$ and $A, B > 0$, then f is identically zero.

(a) If f is even, show that $\hat{f}$ extends to an even entire function. Moreover, if $g(z) = \hat{f}(z^{1/2})$, then g satisfies

$$|g(x)| \leq ce^{-\pi x} \quad \text{and} \quad |g(z)| \leq ce^{\pi R \sin^2(\theta/2)} \leq ce^{\pi |z|}$$

when $x \in \mathbb{R}$ and $z = Re^{i\theta}$ with $R \geq 0$ and $\theta \in \mathbb{R}$.

(b) Apply the Phragmén-Lindelöf principle to the function

$$F(z) = g(z)e^{\gamma z} \quad \text{where } \gamma = i\pi \frac{e^{-i\pi/(2\beta)}}{\sin \pi/(2\beta)}$$

and the sector $0 \leq \theta \leq \pi/\beta < \pi$, and let $\beta \to \pi$ to deduce that $e^{\pi z} g(z)$ is bounded in the closed upper half-plane. The same result holds in the lower half-plane, so by Liouville's theorem $e^{\pi z} g(z)$ is constant, as desired.

(c) If f is odd, then $\hat{f}(0) = 0$, and apply the above argument to $\hat{f}(z)/z$ to deduce that $f = \hat{f} = 0$. Finally, write an arbitrary f as an appropriate sum of an even function and an odd function.

5 Problems

1. Suppose $\hat{f}(\xi) = O(e^{-a|\xi|^p})$ as $|\xi| \to \infty$, for some $p > 1$. Then f is holomorphic for all z and satisfies the growth condition

$$|f(z)| \leq Ae^{a|z|^q}$$

where $1/p + 1/q = 1$.

Note that on the one hand, when $p \to \infty$ then $q \to 1$, and this limiting case can be interpreted as part of Theorem 3.3. On the other hand, when $p \to 1$ then $q \to \infty$, and this limiting case in a sense brings us back to Theorem 2.1.

[Hint: To prove the result, use the inequality $-\xi^p + \xi u \leq u^q$, which is valid when ξ and u are non-negative. To establish this inequality, examine separately the cases $\xi^p \geq \xi u$ and $\xi^p < \xi u$; note also that the functions $\xi = u^{q-1}$ and $u = \xi^{p-1}$ are inverses of each other because $(p-1)(q-1) = 1$.]

2. The problem is to solve the differential equation

$$a_n \frac{d^n}{dt^n} u(t) + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} u(t) + \cdots + a_0 u(t) = f(t)\,,$$

where $a_0, a_1, \ldots, a_n$ are complex constants, and f is a given function. Here we suppose that f has bounded support and is smooth (say of class C^2).

(a) Let

$$\hat{f}(z) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i z t}\, dt.$$

Observe that $\hat{f}$ is an entire function, and using integration by parts show that

$$|\hat{f}(x+iy)| \leq \frac{A}{1+x^2}$$

if $|y| \leq a$ for any fixed $a \geq 0$.

(b) Write

$$P(z) = a_n (2\pi i z)^n + a_{n-1} (2\pi i z)^{n-1} + \cdots + a_0.$$

Find a real number c so that $P(z)$ does not vanish on the line

$$L = \{z : z = x + ic, \quad x \in \mathbb{R}\}.$$

(c) Set

$$u(t) = \int_L \frac{e^{2\pi i z t}}{P(z)} \hat{f}(z)\, dz.$$

Check that

$$\sum_{j=0}^{n} a_j \left(\frac{d}{dt}\right)^j u(t) = \int_L e^{2\pi i z t} \hat{f}(z)\, dz$$

and

$$\int_L e^{2\pi i z t} \hat{f}(z)\, dz = \int_{-\infty}^{\infty} e^{2\pi i x t} \hat{f}(x)\, dx.$$

Conclude by the Fourier inversion theorem that

$$\sum_{j=0}^{n} a_j \left(\frac{d}{dt}\right)^j u(t) = f(t).$$

Note that the solution u depends on the choice c.

3.* In this problem, we investigate the behavior of certain bounded holomorphic functions in an infinite strip. The particular result described here is sometimes called the three-lines lemma.

(a) Suppose $F(z)$ is holomorphic and bounded in the strip $0 < \text{Im}(z) < 1$ and continuous on its closure. If $|F(z)| \leq 1$ on the boundary lines, then $|F(z)| \leq 1$ throughout the strip.

(b) For the more general F, let $\sup_{x\in\mathbb{R}} |F(x)| = M_0$ and $\sup_{x\in\mathbb{R}} |F(x+i)| = M_1$. Then,

$$\sup_{x\in\mathbb{R}} |F(x+iy)| \leq M_0^{1-y} M_1^y, \qquad \text{if } 0 \leq y \leq 1.$$

(c) As a consequence, prove that $\log \sup_{x\in\mathbb{R}} |F(x+iy)|$ is a convex function of y when $0 \leq y \leq 1$.

[Hint: For part (a), apply the maximum modulus principle to $F_\epsilon(z) = F(z)e^{-\epsilon z^2}$. For part (b), consider $M_0^{z-1} M_1^{-z} F(z)$.]

4.* There is a relation between the Paley-Wiener theorem and an earlier representation due to E. Borel.

(a) A function $f(z)$, holomorphic for all z, satisfies $|f(z)| \leq A_\epsilon e^{2\pi(M+\epsilon)|z|}$ for all ϵ if and only if it is representable in the form

$$f(z) = \int_C e^{2\pi i z w} g(w)\, dw$$

where g is holomorphic outside the circle of radius M centered at the origin, and g vanishes at infinity. Here C is any circle centered at the origin of radius larger than M. In fact, if $f(z) = \sum a_n z^n$, then $g(w) = \sum_{n=0}^\infty A_n w^{-n-1}$ with $a_n = A_n (2\pi i)^{n+1}/n!$.

(b) The connection with Theorem 3.3 is as follows. For these functions f (for which in addition f and $\hat{f}$ are of moderate decrease on the real axis), one can assert that the g above is holomorphic in the larger region, which consists of the slit plane $\mathbb{C} - [-M, M]$. Moreover, the relation between g and the Fourier transform $\hat{f}$ is

$$g(z) = \frac{1}{2\pi i} \int_{-M}^{M} \frac{\hat{f}(\xi)}{\xi - z}\, d\xi$$

so that $\hat{f}$ represents the jump of g across the segment $[-M, M]$; that is,

$$\hat{f}(x) = \lim_{\epsilon\to 0, \epsilon > 0} g(x+i\epsilon) - g(x - i\epsilon).$$

See Problem 5 in Chapter 3.