Math 185 Final May 9th, 2023

Name: ______________________________

- You have 3 hours to complete the exam.
- Please provides all intermediate steps for calculation problems and justifications for proof based problems.

Good Luck!

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

We use $\mathbb{D} = \{z : |z| < 1\}$ for the open unit disk, $C = \partial\mathbb{D}$ for its boundary, i.e the unit circle, and $\widehat{\mathbb{C}}$ for the extended complex plane $\mathbb{C} \cup \{\infty\}$.

1. (10 points, 2 points each) True or False (1pt). Please provide your reasoning(1pt).

True
(1) If f is a bounded holomorphic function on the punctured disk $\mathbb{D}\backslash\{0\}$ then f can be extended to a holomorphic function on $\mathbb{D}$.

True
(2) Let $f(z)$ be an entire function. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in \mathbb{D}$, then the equality holds for all $z \in \mathbb{C}$.

True
(3) Let $f(z)$ be a holomorphic function on $\mathbb{C}$, and let $g(z) = \int_0^1 f(z+x)dx$. Then $g(z)$ is a holomorphic function.

False
(4) Let $f(z)$ be a holomorphic function on $\mathbb{C}$, and let $g(z) = \int_0^1 f(z+x)dx$. If $g(z) = 0$ for all z, then $f(z)$ has to be 0. $f(z) = e^{2\pi i z}$

True
(5) Let $f(z)$ be a holomorphic function on $\mathbb{C}$. Suppose $f(z) \in \mathbb{R}$ for all $z \in \mathbb{R}$, then $f(\bar{z}) = \overline{f(z)}$ for all $z \in \mathbb{C}$.

2. (10 points) Given that $\sin(z) = (e^{iz} - e^{-iz})/(2i)$ and $\cos(z) = (e^{iz} + e^{-iz})/2$ are holomorphic functions on $\mathbb{C}$, and that for all $x \in \mathbb{R}$, we have $\sin^2(x) + \cos^2(x) = 1$. Prove that for all $z \in \mathbb{C}$, we have $\sin^2(z) + \cos^2(z) = 1$.

either direct check, or say if $f = g$ on $\mathbb{R}$, and f, g hol'c, then $f = g$ on $\mathbb{C}$

let $u = z-5$, then $z = u+5$. $\frac{1}{z-1} = \frac{1}{u+5-1} = \frac{1}{4+u}$

$= \frac{1}{4}\frac{1}{1+u/4}$

3. (10 points) Compute the first 3 terms of the Taylor expansion of $1/(z-1)$ at $z = 5$. What is its the radius of convergence?

$R = 4$ = distance between 1 and 5

$= \frac{1}{4}\left(1 - \frac{u}{4} + \frac{u^2}{4^2}\right) \cdots$

$= \frac{1}{4} - \frac{u^2}{16} + \frac{u^2}{64} \cdots$

4. (10 points) Compute the first 3 terms of the Laurent expansion of e^z/z^2 at $z = 0$. Then, compute the integral

$$\oint_{|z|=1} e^z/z^2 dz.$$

$= 2\pi i \cdot \operatorname{Res}_{z=0}\left(\frac{e^z}{z^2}\right) = 2\pi i$

$\frac{e^z}{z^2} = \frac{1}{z^2}\left(1 + z + \frac{z^2}{2} + \cdots\right)$

$= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \cdots$

5. (10 points) Given any complex number $a \in \mathbb{C}$ with $Re(a) > 0$, compute the integral

$$I_a = \int_0^\infty \frac{a}{1+(ax)^2} dx$$

.

say $z = ax$, $a = e^{i\theta} \cdot r$

$\int_0^{e^{i\theta}\infty} \frac{1}{1+z^2} dz = \int_0^\infty \frac{1}{1+z^2} dz$

$= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{1+z^2} dz$

$= \frac{1}{2}\int \frac{1}{1+z^2} dz$

$= \frac{1}{2} \cdot \frac{2\pi i}{2i}$

$= \frac{\pi}{2}$

6. (10 points) For any $a \in \mathbb{D}$, let $B_a(z) = (z-a)/(1-\bar{a}z)$. Find a point $z \in \mathbb{C}$, such that $B_{1/2}(z) = 2$ (5pt). Find a point $b \in \mathbb{C}$, such that $B_{1/2}(z) \neq b$ for any $z \in \mathbb{C}$. a

$\frac{z - \frac{1}{2}}{1 - \frac{1}{2}z} = 2$

$\Leftrightarrow z - \frac{1}{2} = 2 - z$

$\Leftrightarrow 2z = \frac{5}{2}$

$\Leftrightarrow \boxed{z = \frac{5}{4}}$

set $z = \infty$, then $B_a(\infty) = \frac{1}{-\bar{a}} = -\frac{1}{\frac{1}{2}} = -2$

not achievable

7. (10 points) Let $f(z)$ be a holomorphic function on $\mathbb{D}$. Suppose we know $Re(f) = Im(f)$ on $\mathbb{D}$, prove that f has to be a constant function.

Cauchy-Riemann.

8. (10 points) Prove that for all positive integer n, we have

$$\frac{1}{2\pi i}\oint_{|z|=n+1/2} \frac{\cos(z)}{\sin(z)} dz = 2n+1$$

$\oint \frac{f' dz}{f}$ type.

9. (10 points) Find a holomorphic function $f(z)$ on the right half plane $\{z \mid Re(z) > 0\}$, such that $f(z) = 0$ exactly when $z = 1/n$, where $n = 1, 2, 3\cdots$ (meaning $f(z)$ is nonzero elsewhere).

e.g. $\sin(\pi/z)$.

or, let $u = \frac{1}{z}$, we want zero at $u = n$, then one can do

$\prod_{n=1}^{\infty}\left(1 - \frac{u}{n}\right)e^{\frac{u}{n}}$

$= \prod_{n=1}^{\infty}\left(1 - \frac{1}{nz}\right)e^{\frac{1}{nz}}$

10. (10 points) If f is a holomorphic function on $\mathbb{C}$, such that $f(z) = f(z+1)$ for all z, then prove that for any $a \in \mathbb{C}$, the integral $I_a = \int_{x=0}^1 f(a+x)dx$ is a constant (independent of a).

I_a hol'c, and.

$\frac{d}{da} I_a = \int_0^1 f'(a+x)\,dx$

$= f(a+1) - f(a)$

$= 0$