product expansion for the sine function,

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right) = \pi z\left(1 - z^2\right)\left(1 - \frac{z^2}{4}\right)\cdots. \tag{3.3}$$

Exercises for XIII.3

1. Evaluate the following.

$$\text{(a)} \prod_{n=1}^{\infty}\left(1 + \frac{1}{n(n+2)}\right) \quad \text{(b)} \prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) \quad \text{(c)} \prod_{n=3}^{\infty}\left(\frac{n^2-1}{n^2-4}\right)$$

2. Define $a_k = -\frac{1}{\sqrt{k}}$ if k is odd, and $a_k = \frac{1}{\sqrt{k}} + \frac{1}{k} + \frac{1}{k\sqrt{k}}$ if k is even. Show that $\prod(1 + a_k)$ converges, while $\sum a_k$ and $\sum a_k^2$ diverge.

3. Show that if $t_j \geq 0$, then $\prod(1 + t_j) \leq \exp(\sum t_j)$.

4. Show that if $0 < t_j < 1$, then $\prod(1 - t_j)$ converges if and only if $\sum t_j$ converges.

5. Show that the infinite product $\prod(1 + a_j)$ converges if and only if there is $N \geq 1$ such that $\lim_{m\to\infty}\prod_{j=N}^{m}(1 + a_j)$ exists and is nonzero.

6. Show that $\prod(1 + a_j)$ converges if and only if $\prod_{j=m}^{n}(1 + a_j) \to 1$ as $m, n \to \infty$. *Hint.* Take logarithms and invoke the Cauchy criterion for series.

7. Show that if $\prod(1 + a_k)$ converges, then $\prod|1 + a_k|$ converges.

8. Suppose $a_k \to 0$. Show that the series $\sum a_k$ converges absolutely if and only if both the series $\sum \operatorname{Arg}(1 + a_k)$ and $\sum \operatorname{Log}|1 + a_k|$ converge absolutely.

9. Show that $\displaystyle\prod_{n=1}^{\infty}\left(1 + \frac{1}{n^2}\right) = \frac{e^{\pi} - e^{-\pi}}{2\pi}$.

10. Show that $\displaystyle\prod_{n=0}^{\infty}\left(1 + z^{2^n}\right) = \frac{1}{1-z}$ for $|z| < 1$.

11. Show that if $p_k(z)$ is a polynomial of degree k such that $p_k(0) = 1$ and $p_k(z)$ has no zeros in the disk $\{|z| \leq k^3\}$, then $\prod p_k(z)$ converges normally.

12. Establish one of the following formulae, and deduce from it the other using logarithmic differentiation:

$$e^z - 1 = ze^{z/2} \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 k^2}\right),$$

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + 2z \sum_{k=1}^{\infty} \frac{1}{z^2 + 4\pi^2 k^2}.$$

13. Use the infinite product expansion for $\sin(\pi z)$ to show that the **Wallis product**

$$\prod_{k=1}^{\infty} \frac{(2k)^2}{(2k-1)(2k+1)} = \lim_{n\to\infty} \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7} \cdots \frac{(2n)\cdot(2n)}{(2n-1)\cdot(2n+1)}$$

converges to $\pi/2$. Use this to show that

$$\lim_{n\to\infty} \frac{[n!]^2}{(2n)!} \frac{2^{2n}}{\sqrt{n}} = \sqrt{\pi}.$$

14. Show that if $t > 0$, then $\prod_{-m \le k \le tm} \left(1 + \frac{z}{k}\right)$ converges to $\frac{\sin(\pi z)}{\pi z} t^z$ as $m \to \infty$.

15. Show that $\frac{1}{z} \prod_{n=1}^{\infty} \frac{n}{z+n} \left(\frac{n+1}{n}\right)^z$ converges to a meromorphic function $\Gamma(z)$ whose poles are simple poles at 0 and the negative integers. Show that

$$\Gamma(z) = \lim_{m\to\infty} \frac{(m-1)!\, m^z}{z(z+1)\cdots(z+m-1)}.$$

Show that $\Gamma(z+1) = z\Gamma(z)$. Show that $\Gamma(n+1) = n!$ for positive integers n. *Remark.* The function $\Gamma(z)$ is called the **gamma function**. It was first introduced by Euler, who defined it to be the limit above. We will give an equivalent definition in the next chapter.

16. Let α_k be a sequence of complex numbers, with possible repetitions, such that $|\alpha_k| < 1$ and $|\alpha_k| \to 1$, and consider the **infinite Blaschke product** defined by

$$B(z) = \prod \frac{\overline{\alpha_k}}{|\alpha_k|} \frac{\alpha_k - z}{1 - \overline{\alpha_k} z},$$

where the factors corresponding to $\alpha_k = 0$ are z.

(a) Suppose that $\sum(1 - |\alpha_k|) < \infty$. Let E be the set of accumulation points on the unit circle $\partial\mathbb{D}$ of the α_k's. Show that the infinite product converges normally on $\mathbb{C}^*\backslash E$ to a meromorphic function $B(z)$, with the following properties: $|B(z)| < 1$ for $z \in \mathbb{D}$, $|B(z)| = 1$ for $z \in \partial\mathbb{D}\backslash E$, and $B(z)$ has zeros precisely at the points α_k.

(b) Show that if $\sum(1 - |\alpha_k|) = +\infty$, then the partial products converge uniformly on compact subsets of $\mathbb{D}$ to 0.

(c) Suppose that $f(z)$ is a bounded analytic function on $\mathbb{D}$ that is not identically zero. Show that $f(z)$ has a factorization $f(z) = B(z)g(z)$, where $B(z)$ is a (finite or infinite) Blaschke product, and $g(z)$ is a bounded analytic function on $\mathbb{D}$ with no zeros. In particular, the zeros $\alpha_1, \alpha_2, \ldots$ of $f(z)$, repeated according to multiplicity, satisfy $\sum(1 - |\alpha_k|) < +\infty$.

4. The Weierstrass Product Theorem

The Weierstrass product theorem is a companion theorem to the Mittag-Leffler theorem. The Mittag-Leffler theorem asserts that we can prescribe the poles and principal parts of a meromorphic function. The Weierstrass product theorem asserts that we can prescribe the zeros and poles of a meromorphic function together with their orders.

Recall that the order of a meromorphic function $f(z)$ at a point z_0 is the order of the zero if $f(z_0) = 0$, and it is minus the order of the pole if $f(z)$ has a pole at z_0. If z_0 is neither a pole nor a zero of $f(z)$, the order of $f(z)$ at z_0 is defined to be 0.

Theorem (Weierstrass Product Theorem). *Let D be a domain in the complex plane. Let $\{z_k\}$ be a sequence of distinct points of D with no accumulation point in D, and let $\{n_k\}$ be a sequence of integers (positive or negative). Then there is a meromorphic function $f(z)$ on D whose only zeros and poles are at the points z_k, such that the order of $f(z)$ at z_k is n_k.*

The proof runs parallel to the proof of the Mittag-Leffler theorem. Let K_m be the set of $z \in D$ such that $|z| \leq m$ and the distance from z to ∂D is at least $1/m$. Then K_m is a compact subset of D, $K_m \subset K_{m+1}$, and each component of $\mathbb{C}^*\backslash K_m$ contains a point of $\mathbb{C}^*\backslash D$. Suppose $z_k \in K_{m+1}\backslash K_m$. We connect z_k to a point $w_k \in C^*\backslash D$ by a simple curve γ_k in $\mathbb{C}^*\backslash K_m$. If $w_k \neq \infty$ we define $f_k(z)$ to be an analytic branch of $\log((z - z_k)/(z - w_k))$ in the simply connected domain $\mathbb{C}^*\backslash\gamma_k$. If $w_k = \infty$, we take $f_k(z)$ to be an analytic branch of $\log(1 - z/z_k)$. By Runge's theorem, there is a rational function $g_k(z)$ with only pole at w_k such that $|f_k(z) - g_k(z)| \leq 2^{-k}/n_k$ on K_m. We consider the product

$$f(z) = \prod_{k=1}^{\infty} \left(\frac{z - z_k}{z - w_k}\right)^{n_k} e^{-n_k g_k(z)},$$