Exercises for XIII.4

1. Let $G(z)$ be the entire function defined by the infinite product (4.1). Show that $\pi z G(z)G(-z) = \sin(\pi z)$.

2. Construct an entire function that has simple zeros at the points n^2, $n \geq 0$, and no other zeros.

3. Construct an entire function that has simple zeros on the real axis at the points $\pm n^{1/4}$, $n \geq 0$, and no other zeros.

4. Construct an entire function that has simple zeros on the positive real axis at the points $\sqrt{n}$, $n \geq 1$, and double zeros on the imaginary axis at the points $\pm i\sqrt{n}$, $n \geq 1$, and no other zeros.

5. Construct an entire function that has simple zeros at the Gaussian integers $m + ni$, $-\infty < m, n < \infty$, and no other zeros.

6. Find *all* entire functions $f(z)$ that satisfy the functional equation $f(2z) = (1 - 2z)f(z)$. Express the answer in terms of an infinite product.

7. Show that
$$\left|\operatorname{Log}(1-\zeta) + \zeta + \frac{\zeta^2}{2} + \frac{\zeta^3}{3} + \cdots + \frac{\zeta^N}{N}\right| \leq \frac{1}{N}\frac{|\zeta|^{N+1}}{1-|\zeta|},$$
for $|\zeta| < 1$.

8. Let $\{z_k\}$ be a sequence of distinct nonzero points such that $|z_k| \rightarrow \infty$, Let $N \geq 0$, and let $\{m_k\}$ be a sequence of positive integers such that $\sum m_k |z_k|^{-N-1} < \infty$. Show that
$$\prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right)^{m_k} \exp\left\{m_k \left[\frac{z}{z_k} + \frac{z^2}{2z_k^2} + \cdots + \frac{z^N}{N z_k^N}\right]\right\}$$
converges normally to an entire function with zeros of order m_k at z_k and no other zeros. *Hint.* See the estimate in the preceding exercise.

9. Show that any meromorphic function $f(z)$ on a domain D is the quotient $f(z) = g(z)/h(z)$ of two analytic functions $g(z)$ and $h(z)$ on D.

10. Let $f(z)$ be a meromorphic function on a simply connected domain D. Show that the meromorphic functions with the same zeros and poles of the same orders as $f(z)$ are precisely the functions of the form $f(z)e^{h(z)}$, where $h(z)$ is analytic on D.

11. Give a brief solution of Exercise 1.5 on interpolating sequences based on the Mittag-Leffler theorem and the Weierstrass product theorem.